Methods in Applied Mathematics (MATHS 362)
Lecture Notes

JP-E

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Contents

1 Lecture 1: Introduction

1.1 A simple example [1, §1.1] and [2, §1.1]

The motion of an object projected radially upward from the surface of the Earth is given by the equation

$$\frac{d^2x}{dt^2} = -\frac{gR^2}{(x+R)^2}, \qquad \text{for } t > 0, \tag{1}$$

where $x(t)$ is the height of the object measured from the surface of the Earth, R is the radius of the Earth, and g is the gravitational acceleration at the surface. We assume the object starts from the surface with a given upward velocity, so the initial conditions are

$$x(0) = 0, \quad \frac{dx}{dt}(0) = v_0,$$

where v_0 is positive.

Solving (1) is challenging because it involves a second-order nonlinear ODE. One option for finding the solution is to use numerical methods and a computer. However, this approach does not provide much insight into how the solution depends on the terms in the equation. We are interested in obtaining at least an approximate solution of this problem.

Typically, it is assumed that R is significantly larger than even the largest value of $x(t)$ (so that $x(t) \ll R$). If true, then we should be able to replace $R + x$ with R in the denominator of the right-hand side of (1) to obtain an approximate solution. In this case, the problem reduces to solving the simpler ODE

$$\frac{d^2x}{dt^2} = -g, \qquad \text{for } t > 0, \tag{2}$$

with the same initial conditions. The solution to (2) is

$$x(t) = v_0 t - \frac{1}{2}gt^2.$$

The solution is shown schematically in Figure 1.

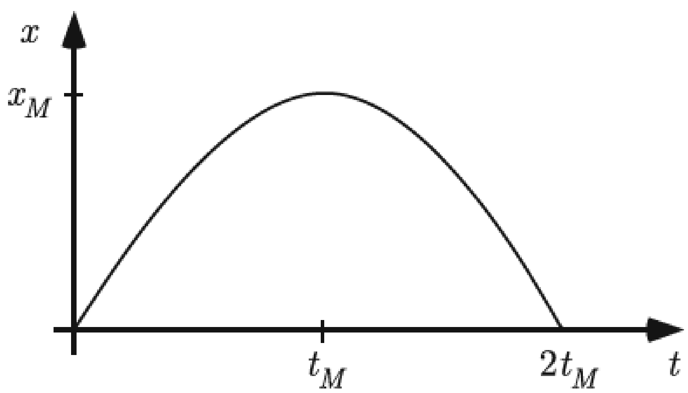


Figure 1: Schematic of the solution to the projectile motion problem.

There are some critiques of this approach.

Observations:

- The reduction from (1) to (2) is heuristic. We have not justified the approximation in a rigorous way.
- We discarded the dependence on x in the denominator of the right-hand side of (1). Likewise, we do not know the effects of the nonlinearity.
- We do not know how to estimate the error of the approximation, or how to improve the approximation if we want a better one.

The first part of the course is about asymptotic analysis. The goal is to develop techniques to make the reduction process more systematic. First, we need to understand what is 'large' or 'small' in a problem and then construct increasingly accurate approximations to the solution.

1.2 Recap of background concepts [2, §1.2, 1.3]

We will need to use some background concepts in the course. These include Taylor's theorem, L'Hôpital's rule, and orders of magnitude.

1.2.1 Taylor's theorem

Taylor's Theorem:

Given a function $f(\varepsilon)$, suppose its $(n+1)$-th derivative $f^{(n+1)}(\varepsilon)$ exists and is continuous for $\varepsilon_a < \varepsilon < \varepsilon_b$. If ε_0 and ε are points in the interval $(\varepsilon_a, \varepsilon_b)$, then

$$f(\varepsilon) = f(\varepsilon_0) + f'(\varepsilon_0)(\varepsilon - \varepsilon_0) + \cdots + \frac{f^{(n)}(\varepsilon_0)}{n!}(\varepsilon - \varepsilon_0)^n + R_{n+1}, \tag{3}$$

where

$$R_{n+1} = \frac{f^{(n+1)}(\xi)}{(n+1)!}(\varepsilon - \varepsilon_0)^{n+1}, \tag{4}$$

and ξ is some point between ε_0 and ε.

Theorem 1.2.1 is useful because it allows us to approximate a function $f(\varepsilon)$ by a polynomial in ε around some point ε_0. The error of the approximation is given by the remainder term R_{n+1}.

Example:

Find the first three terms in the Taylor expansion of $f(\varepsilon) = \sin(e^\varepsilon)$ for $\varepsilon_0 = 0$.

Solution: We compute the derivatives.

$$\begin{aligned} f'(\varepsilon) &= \cos(e^\varepsilon)\ e^\varepsilon, \\ f''(\varepsilon) &= -e^{2\varepsilon}\sin(e^\varepsilon) + \cos(e^\varepsilon)\ e^\varepsilon. \end{aligned}$$

Substituting $\varepsilon_0 = 0$ into the function and its derivatives, we obtain

$$\begin{aligned} f(0) &= \sin(1), \\ f'(0) &= \cos(1), \\ f''(0) &= -\sin(1) + \cos(1). \end{aligned}$$

Thus, the first three terms in the Taylor expansion of $f(\varepsilon)$ around $\varepsilon_0 = 0$ are

$$f(\varepsilon) = \sin(1) + \cos(1)\,\varepsilon + \frac{-\sin(1) + \cos(1)}{2}\,\varepsilon^2 + \cdots .$$

Exercise:

Find the first three terms in the Taylor expansion of

$$f(\varepsilon) = \frac{e^\varepsilon}{1-\varepsilon}$$

around $\varepsilon_0 = 0$.

Solution: We compute the derivatives of f.

$$f(\varepsilon) = \frac{e^\varepsilon}{1-\varepsilon}.$$

Using the quotient rule,

$$f'(\varepsilon) = \frac{e^\varepsilon(1-\varepsilon) + e^\varepsilon}{(1-\varepsilon)^2} = \frac{e^\varepsilon(2-\varepsilon)}{(1-\varepsilon)^2}.$$

Differentiating again,

$$f''(\varepsilon) = \frac{d}{d\varepsilon}\left(\frac{e^\varepsilon(2-\varepsilon)}{(1-\varepsilon)^2}\right) = \frac{e^\varepsilon\left(\varepsilon^2 - 4\varepsilon + 5\right)}{(1-\varepsilon)^3}.$$

Evaluating at $\varepsilon = 0$,

$$f(0) = 1, \qquad f'(0) = 2, \qquad f''(0) = 5.$$

Hence, the Taylor expansion of f around 0 begins with

$$f(\varepsilon) = 1 + 2\varepsilon + \frac{5}{2}\varepsilon^2 + \cdots .$$

1.2.2 L'Hôpital's Rule

L'Hôpital's Rule:

Let f and g be differentiable on an open interval containing ε_0, except possibly at ε_0 itself. Assume that

$$\lim_{\varepsilon \to \varepsilon_0} f(\varepsilon) = 0 \quad \text{and} \quad \lim_{\varepsilon \to \varepsilon_0} g(\varepsilon) = 0,$$

or
$$\lim_{\varepsilon \to \varepsilon_0} f(\varepsilon) = \pm\infty \quad \text{and} \quad \lim_{\varepsilon \to \varepsilon_0} g(\varepsilon) = \pm\infty.$$
Suppose moreover that $g'(\varepsilon) \neq 0$ in a punctured neighbourhood of ε_0 and that the limit
$$\lim_{\varepsilon \to \varepsilon_0} \frac{f'(\varepsilon)}{g'(\varepsilon)}$$
exists (finite or infinite). Then
$$\lim_{\varepsilon \to \varepsilon_0} \frac{f(\varepsilon)}{g(\varepsilon)} = \lim_{\varepsilon \to \varepsilon_0} \frac{f'(\varepsilon)}{g'(\varepsilon)}.$$

Example:

Use L'Hôpital's Rule to evaluate
$$\lim_{\varepsilon \to 0} \frac{e^{\varepsilon} - 1}{\varepsilon}.$$
Solution: As $\varepsilon \to 0$, we have
$$e^{\varepsilon} - 1 \to 0 \quad \text{and} \quad \varepsilon \to 0,$$
so the limit is of the indeterminate form $\frac{0}{0}$. We may therefore apply L'Hôpital's Rule. Differentiate the numerator and denominator:
$$\frac{d}{d\varepsilon}(e^{\varepsilon} - 1) = e^{\varepsilon}, \qquad \frac{d}{d\varepsilon}(\varepsilon) = 1.$$
Hence,
$$\lim_{\varepsilon \to 0} \frac{e^{\varepsilon} - 1}{\varepsilon} = \lim_{\varepsilon \to 0} e^{\varepsilon} = e^0 = 1.$$

1.2.3 Order Symbols

Note: Throughout, we consider $\varepsilon > 0$, so the limit $\varepsilon \downarrow \varepsilon_0$ is a one-sided limit, that represents the limit as ε approaches ε_0 from above. If $\varepsilon_0 = 0$, then $\varepsilon \downarrow 0$ means $\varepsilon \to 0^+$.

We are interested in how functions behave as a parameter, typically ε, becomes small. For example, the function $f(\varepsilon) = \varepsilon$ does not converge to zero as fast as $g(\varepsilon) = \varepsilon^2$ as $\varepsilon \to 0$. To capture this idea, we introduce the following notation.

Big O and little o notation:

Let $f(\varepsilon)$ and $g(\varepsilon)$ be functions defined for
$$\varepsilon_0 - \delta < \varepsilon < \varepsilon_0 + \delta, \qquad \varepsilon \neq \varepsilon_0,$$
for some $\delta > 0$, and assume that $g(\varepsilon) \neq 0$ there.
We say that $f(\varepsilon)$ is *big O* of $g(\varepsilon)$ as $\varepsilon \downarrow \varepsilon_0$, and write
$$f(\varepsilon) = O(g(\varepsilon)) \quad \text{as } \varepsilon \downarrow \varepsilon_0,$$

if there exist positive constants C and δ such that

$$|f(\varepsilon)| \leq C\,|g(\varepsilon)| \quad \text{whenever } 0 < |\varepsilon - \varepsilon_0| < \delta.$$

We say that $f(\varepsilon)$ is *little o* of $g(\varepsilon)$ as $\varepsilon \downarrow \varepsilon_0$, and write

$$f(\varepsilon) = o(g(\varepsilon)) \quad \text{as } \varepsilon \downarrow \varepsilon_0,$$

if

$$\lim_{\varepsilon \downarrow \varepsilon_0} \frac{f(\varepsilon)}{g(\varepsilon)} = 0.$$

Sometimes, we will use the following result:

Theorem:

If

$$\lim_{\varepsilon \downarrow \varepsilon_0} \frac{f(\varepsilon)}{g(\varepsilon)} = L,$$

where L is a finite constant, then

$$f(\varepsilon) = O(g(\varepsilon)) \quad \text{as } \varepsilon \downarrow \varepsilon_0.$$

Note: If one wished to be completely precise, one could introduce a refined notation, for example writing

$$f(\varepsilon) = O_s(\cdot)$$

to mean that $f(\varepsilon)$ is $O(\cdot)$ but not $o(\cdot)$; that is, $f = O(\cdot)$ and $f \not\to 0$.

Example:

Suppose that $f(\varepsilon) = \varepsilon^2$, $g(\varepsilon) = \varepsilon$, and $h(\varepsilon) = -3\varepsilon^2 + 5\varepsilon^6$. Then

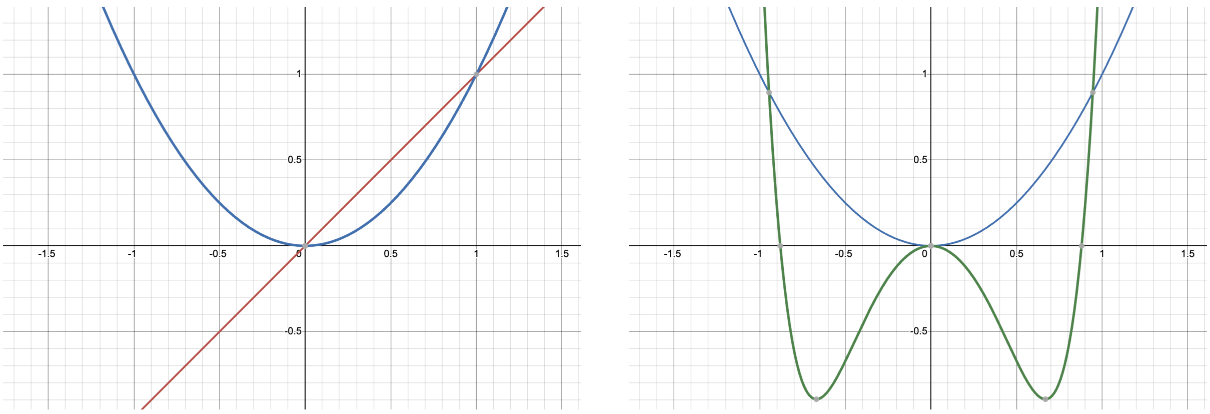
$$\lim_{\varepsilon \downarrow 0} \frac{f(\varepsilon)}{g(\varepsilon)} = \lim_{\varepsilon \downarrow 0} \frac{\varepsilon^2}{\varepsilon} = \lim_{\varepsilon \downarrow 0} \varepsilon = 0,$$

so $f(\varepsilon) = o(g(\varepsilon))$ as $\varepsilon \downarrow 0$.
Moreover,

$$\lim_{\varepsilon \downarrow 0} \frac{f(\varepsilon)}{h(\varepsilon)} = \lim_{\varepsilon \downarrow 0} \frac{\varepsilon^2}{-3\varepsilon^2 + 5\varepsilon^6} = \lim_{\varepsilon \downarrow 0} \frac{1}{-3 + 5\varepsilon^4} = -\frac{1}{3}.$$

Since this limit exists and is finite, it follows that $f(\varepsilon) = O(h(\varepsilon))$ as $\varepsilon \downarrow 0$.

Interpretation: As ε approaches 0, the function $g(\varepsilon) = \varepsilon$ goes to 0 *more slowly* than $f(\varepsilon) = \varepsilon^2$. In other words, ε^2 becomes negligible compared to ε near 0.
On the other hand, the functions $f(\varepsilon) = \varepsilon^2$ and $h(\varepsilon) = -3\varepsilon^2 + 5\varepsilon^6$ vanish at essentially the *same rate*, since their ratio tends to a nonzero constant. This means that, close to 0, the dominant term in $h(\varepsilon)$ is $-3\varepsilon^2$, and the higher-order term $5\varepsilon^6$ is negligible. Hence, $h(\varepsilon)$ behaves like $-3\varepsilon^2$ for small ε.



(a) Graph of the functions $f(\varepsilon) = \varepsilon^2$ and $g(\varepsilon) = \varepsilon$.

(b) Graph of the functions $f(\varepsilon) = \varepsilon^2$ and $h(\varepsilon) = -3\varepsilon^2 + 5\varepsilon^6$.

Figure 2: Examples of big O and little o notation.

Exercise:

Show that if $f(\varepsilon) = \varepsilon \sin\left(1 + \frac{1}{\varepsilon}\right)$ and $g(\varepsilon) = \varepsilon$, then $f(\varepsilon) = O(g(\varepsilon))$ as $\varepsilon \downarrow 0$ but $f(\varepsilon) \neq o(g(\varepsilon))$ as $\varepsilon \downarrow 0$.

Solution: For $\varepsilon \neq 0$ we have

$$\frac{f(\varepsilon)}{g(\varepsilon)} = \sin\left(1 + \frac{1}{\varepsilon}\right).$$

This limit does not exist as $\varepsilon \downarrow 0$, since $\sin\left(1 + \frac{1}{\varepsilon}\right)$ oscillates between -1 and 1. Nevertheless,

$$\left|\frac{f(\varepsilon)}{g(\varepsilon)}\right| = \left|\sin\left(1 + \frac{1}{\varepsilon}\right)\right| \leq 1 \qquad \text{for all } \varepsilon \neq 0.$$

Thus, $|f(\varepsilon)| \leq 1 \cdot |g(\varepsilon)|$ for all $\varepsilon \neq 0$, so $f(\varepsilon) = O(g(\varepsilon))$ as $\varepsilon \downarrow 0$.
Finally, if $f(\varepsilon) = o(g(\varepsilon))$ as $\varepsilon \downarrow 0$, we would have

$$\lim_{\varepsilon \downarrow 0} \frac{f(\varepsilon)}{g(\varepsilon)} = 0,$$

but this limit does not exist (in particular, it is not 0). Hence, $f(\varepsilon) \neq o(g(\varepsilon))$ as $\varepsilon \downarrow 0$.

We will occasionally use the symbols $\ll$ and $\approx$. When we say that $f(\varepsilon) \ll g(\varepsilon)$ as $\varepsilon \downarrow \varepsilon_0$, we mean that $f(\varepsilon) = o(g(\varepsilon))$ as $\varepsilon \downarrow \varepsilon_0$. The symbol $\approx$ is used simply to designate an approximate numerical value, for example, $\pi \approx 3.14159$.

2 Lecture 2: Asymptotic Approximations

The goal of the first method in the course is to construct approximations to the solutions of differential equations. We first need to state what we mean by an approximation.

Suppose we are interested in finding an approximation of $f(\varepsilon) = \varepsilon^2 + \varepsilon^5$ for small ε. Because $\varepsilon^5 \ll \varepsilon^2$ as $\varepsilon \downarrow 0$, we can approximate $f(\varepsilon) \approx \varepsilon^2$. On the other hand, a bad approximation is $f(\varepsilon) \approx \frac{2}{3}\varepsilon^2$, even though the error

$$f(\varepsilon) - \frac{2}{3}\varepsilon^2 = \frac{1}{3}\varepsilon^2 + \varepsilon^5$$

goes to zero as $\varepsilon \downarrow 0$. The problem with this approximation is that the error is of the same order as the approximation itself, since the leading part of the error, $\frac{1}{3}\varepsilon^2$, is of the same order as the approximation $\frac{2}{3}\varepsilon^2$.

Asymptotic approximation:

Given $f(\varepsilon)$ and $g(\varepsilon)$, we say that $g(\varepsilon)$ is an *asymptotic approximation* of $f(\varepsilon)$ as $\varepsilon \downarrow \varepsilon_0$ whenever

$$f(\varepsilon) = g(\varepsilon) + o(g(\varepsilon)) \quad \text{as } \varepsilon \downarrow \varepsilon_0.$$

In this case we write $f \sim g$ as $\varepsilon \downarrow \varepsilon_0$.

In other words, $g(\varepsilon)$ is an asymptotic approximation of $f(\varepsilon)$ as $\varepsilon \downarrow \varepsilon_0$ if the error $f(\varepsilon) - g(\varepsilon)$ is negligible compared to $g(\varepsilon)$ as $\varepsilon \downarrow \varepsilon_0$. In the case where $g(\varepsilon) \neq 0$ near ε_0, this is equivalent to saying that $f \sim g$ as $\varepsilon \downarrow \varepsilon_0$ if

$$\lim_{\varepsilon \downarrow \varepsilon_0} \frac{f(\varepsilon)}{g(\varepsilon)} = 1.$$

Example:

Suppose $f = \sin(\varepsilon)$ and $\varepsilon_0 = 0$. We can obtain an approximation of f by using Taylor's theorem:

$$\sin(\varepsilon) = \varepsilon - \frac{\varepsilon^3}{6} + \frac{\varepsilon^5}{120}\cos(\xi),$$

for some ξ between 0 and ε. Since $\varepsilon^3 \ll \varepsilon$ as $\varepsilon \downarrow 0$, we have

$$\sin(\varepsilon) \sim \varepsilon \quad \text{as } \varepsilon \downarrow 0.$$

Also, since $\varepsilon^5 \ll \varepsilon^3$ as $\varepsilon \downarrow 0$, we have

$$\sin(\varepsilon) \sim \varepsilon - \frac{\varepsilon^3}{6} \quad \text{as } \varepsilon \downarrow 0.$$

Observation:

Note that $\sin(\varepsilon) \sim \varepsilon + 2\varepsilon^2$ as $\varepsilon \downarrow 0$. The definition of asymptotic approximation says little about the comparative accuracy of approximations.

Example:

Suppose $f(\varepsilon) = \alpha + e^{-\alpha/\varepsilon}$ for some constant $0 < \alpha < 1$. Then

$$f(\varepsilon) \sim \alpha \quad \text{as } \varepsilon \downarrow 0,$$

since $e^{-\alpha/\varepsilon} \ll \alpha$ as $\varepsilon \downarrow 0$. How well does this approximate the function for $0 < \alpha < 1$? See Figure 3.

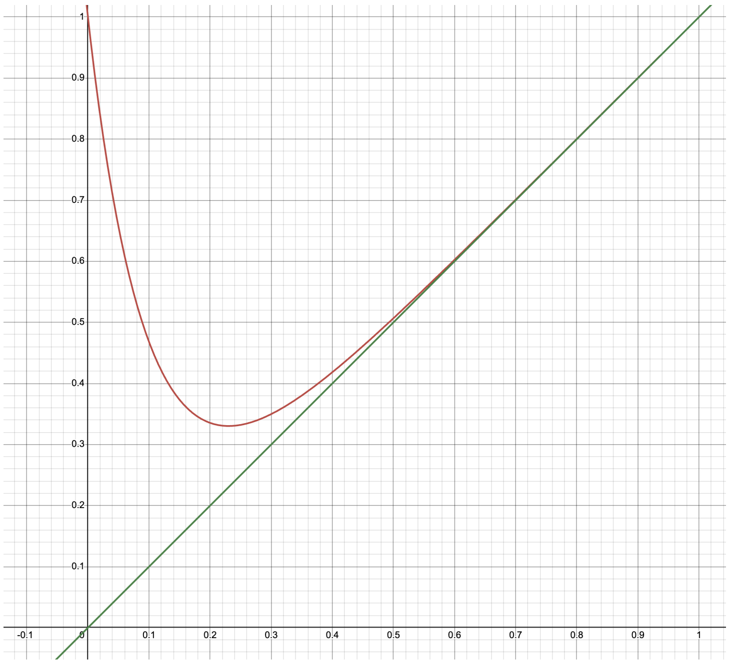


Figure 3: Example of asymptotic approximation for $f(\varepsilon) = \alpha + e^{-\alpha/\varepsilon}$. Fixed $\varepsilon = 0.1$ and the horizontal axis is α.

Exercise:

Consider the function $f(\varepsilon) = \sin(\pi x) + \varepsilon^3$ for some constant $0 \leq x \leq 1$. Is $f \sim \sin(\pi x)$ as $\varepsilon \downarrow 0$ for all x? If not, for which values of x is this true?

2.1 Asymptotic Expansions

Two observations:

- An asymptotic approximation is not unique.
- An asymptotic approximation does not necessarily give a good numerical approximation to the function.

We have been using the sequence $1, \varepsilon, \varepsilon^2, \varepsilon^3, \ldots$ coming from Taylor's theorem to construct asymptotic approximations. There are other sequences that also arise.

Asymptotic sequence:

The functions $\phi_1(\varepsilon), \phi_2(\varepsilon), \ldots$ form an *asymptotic sequence* as $\varepsilon \downarrow \varepsilon_0$ if

$$\phi_{m+1}(\varepsilon) = o(\phi_m(\varepsilon)) \quad \text{as } \varepsilon \downarrow \varepsilon_0$$

for all m.

Asymptotic expansion:

Given a function $f(\varepsilon)$ and an asymptotic sequence $\{\phi_m(\varepsilon)\}$, we say that $f(\varepsilon)$ has an *asymptotic expansion* to n terms with respect to $\{\phi_m(\varepsilon)\}$ if and only if

$$f(\varepsilon) = \sum_{k=1}^{m} a_k \phi_k(\varepsilon) + o(\phi_m(\varepsilon)) \quad \text{as } \varepsilon \downarrow \varepsilon_0, \qquad \text{for each } m = 1, 2, \ldots, n,$$

where the coefficients a_k are independent of ε. In this case we write

$$f(\varepsilon) \sim a_1\phi_1(\varepsilon) + a_2\phi_2(\varepsilon) + \cdots + a_n\phi_n(\varepsilon) \quad \text{as } \varepsilon \downarrow \varepsilon_0.$$

The $\phi_k(\varepsilon)$ are called the scale (or gauge, or basis) functions of the expansion.

Example of (commonly used) scale functions:

- $\phi_1(\varepsilon) = (\varepsilon - \varepsilon_0)^\alpha$, $\phi_2(\varepsilon) = (\varepsilon - \varepsilon_0)^\beta$, $\phi_3(\varepsilon) = (\varepsilon - \varepsilon_0)^\gamma$, ... where $\alpha < \beta < \gamma < \cdots$ are real numbers.
- $\phi_1(\varepsilon) = 1$, $\phi_2(\varepsilon) = e^{-1/\varepsilon}$, $\phi_3(\varepsilon) = e^{-2/\varepsilon}$, $\phi_4(\varepsilon) = e^{-3/\varepsilon}$, ... (typically used as $\varepsilon \downarrow 0$).

How do we find an asymptotic expansion of a function? Normally we use Taylor's theorem, L'Hôpital's rule, or other techniques depending on the form of the function.

Example of an asymptotic expansion:

Taylor's theorem is particularly useful because if the function is smooth enough near ε_0, then the **Taylor expansion** of the function around ε_0 is an asymptotic expansion of the function as $\varepsilon \downarrow \varepsilon_0$ with respect to the sequence $1, \varepsilon - \varepsilon_0, (\varepsilon - \varepsilon_0)^2, \ldots$. Moreover, Taylor's theorem enables us to analyse the error very easily.

Observation:

Every Taylor *expansion* (to any finite order) is an asymptotic expansion, but not every asymptotic expansion is a Taylor series.

Since we will be using Taylor's theorem often throughout, we can use Appendix 17 to store the Taylor expansions of commonly used functions, such as e^ε, $\sin(\varepsilon)$, $\cos(\varepsilon)$, $\ln(1+\varepsilon)$, etc.

Example:

To find a three-term expansion of $f(\varepsilon) = e^\varepsilon$ as $\varepsilon \downarrow 0$, use the Taylor series:

$$e^\varepsilon = 1 + \varepsilon + \frac{\varepsilon^2}{2} + O(\varepsilon^3).$$

Hence,

$$e^\varepsilon \sim 1 + \varepsilon + \frac{\varepsilon^2}{2} \qquad (\varepsilon \downarrow 0).$$

Example:

To find the first two nonzero terms in the expansion of $f(\varepsilon) = \cos(\varepsilon)/\varepsilon$ as $\varepsilon \downarrow 0$, use

$$\cos\varepsilon = 1 - \frac{\varepsilon^2}{2} + O(\varepsilon^4).$$

Thus,

$$\frac{\cos\varepsilon}{\varepsilon} = \frac{1}{\varepsilon} - \frac{\varepsilon}{2} + O(\varepsilon^3), \qquad \varepsilon \downarrow 0.$$

Exercise:

Find a two-term expansion of

$$f(\varepsilon) = \frac{\sqrt{1+\varepsilon}}{\sin(\sqrt{\varepsilon})}$$

around $\varepsilon_0 = 0$.

Solution. Let $t = \sqrt{\varepsilon}$, so that $\varepsilon = t^2$. (Here we take $\varepsilon \downarrow 0$ so that t is real.) Then

$$f(\varepsilon) = \frac{\sqrt{1+t^2}}{\sin t}.$$

Using the Taylor expansions about $t = 0$,

$$\sqrt{1+t^2} = 1 + \frac{t^2}{2} + O(t^4), \qquad \sin t = t - \frac{t^3}{6} + O(t^5) = t\left(1 - \frac{t^2}{6} + O(t^4)\right),$$

we obtain

$$\frac{\sqrt{1+t^2}}{\sin t} = \frac{1 + \frac{t^2}{2} + O(t^4)}{t\left(1 - \frac{t^2}{6} + O(t^4)\right)}.$$

Expanding the reciprocal factor,

$$\frac{1}{1 - \frac{t^2}{6} + O(t^4)} = 1 + \frac{t^2}{6} + O(t^4),$$

and hence

$$\frac{\sqrt{1+t^2}}{\sin t} = \frac{1}{t}\left(1 + \frac{t^2}{2}\right)\left(1 + \frac{t^2}{6}\right) + O(t^3).$$

Multiplying the brackets yields

$$\frac{\sqrt{1+t^2}}{\sin t} = \frac{1}{t}\left(1 + \frac{2}{3}t^2\right) + O(t^3) = \frac{1}{t} + \frac{2}{3}t + O(t^3).$$

Returning to ε, we conclude that

$$\boxed{\frac{\sqrt{1+\varepsilon}}{\sin(\sqrt{\varepsilon})} = \frac{1}{\sqrt{\varepsilon}} + \frac{2}{3}\sqrt{\varepsilon} + O\left(\varepsilon^{3/2}\right), \qquad \varepsilon \downarrow 0.}$$

If we know in advance the scale functions of the expansion, we can find the coefficients of the expansion by using L'Hôpital's rule. That is, if the function has the form

$$f(\varepsilon) \sim a_1\phi_1(\varepsilon) + a_2\phi_2(\varepsilon) + \cdots ,$$

then by definition this means that

$$f(\varepsilon) = a_1\phi_1(\varepsilon) + o(\phi_1(\varepsilon)) \quad \text{as } \varepsilon \downarrow \varepsilon_0.$$

Assuming we can divide by $\phi_1(\varepsilon)$ near ε_0, we get

$$\frac{f(\varepsilon)}{\phi_1(\varepsilon)} = a_1 + o(1) \quad \text{as } \varepsilon \downarrow \varepsilon_0.$$

Hence,

$$a_1 = \lim_{\varepsilon\downarrow\varepsilon_0} \frac{f(\varepsilon)}{\phi_1(\varepsilon)}.$$

Similarly, we can find a_2 by using the fact that

$$f(\varepsilon) - a_1\phi_1(\varepsilon) = a_2\phi_2(\varepsilon) + o(\phi_2(\varepsilon)) \quad \text{as } \varepsilon \downarrow \varepsilon_0,$$

hence

$$a_2 = \lim_{\varepsilon\downarrow\varepsilon_0} \frac{f(\varepsilon) - a_1\phi_1(\varepsilon)}{\phi_2(\varepsilon)},$$

and so on.

This assumes that the scale functions are non-zero for ε near ε_0 and that each of the limits exists.

Example:

Suppose $\phi_1(\varepsilon) = 1$, $\phi_2(\varepsilon) = \varepsilon$, $\phi_3(\varepsilon) = \varepsilon^2, \ldots$, and

$$f(\varepsilon) = \frac{1}{1+\varepsilon} + e^{-1/\varepsilon}.$$

We want to find the coefficients of the asymptotic expansion as $\varepsilon \downarrow 0$.
First,

$$a_1 = \lim_{\varepsilon\downarrow 0} \frac{f(\varepsilon)}{\phi_1(\varepsilon)} = \lim_{\varepsilon\downarrow 0} \left(\frac{1}{1+\varepsilon} + e^{-1/\varepsilon} \right) = 1.$$

Next,

$$\begin{aligned} a_2 &= \lim_{\varepsilon\downarrow 0} \frac{f(\varepsilon) - a_1\phi_1(\varepsilon)}{\phi_2(\varepsilon)} \\ &= \lim_{\varepsilon\downarrow 0} \left[\frac{\frac{1}{1+\varepsilon} - 1}{\varepsilon} + \frac{e^{-1/\varepsilon}}{\varepsilon} \right] = -1, \end{aligned}$$

since $e^{-1/\varepsilon}/\varepsilon \downarrow 0$ as $\varepsilon \downarrow 0$.
Finally,

$$\begin{aligned} a_3 &= \lim_{\varepsilon\downarrow 0} \frac{f(\varepsilon) - a_1\phi_1(\varepsilon) - a_2\phi_2(\varepsilon)}{\phi_3(\varepsilon)} \\ &= \lim_{\varepsilon\downarrow 0} \left[\frac{\frac{1}{1+\varepsilon} - 1 + \varepsilon}{\varepsilon^2} + \frac{e^{-1/\varepsilon}}{\varepsilon^2} \right] = 1, \end{aligned}$$

since $e^{-1/\varepsilon}/\varepsilon^2 \downarrow 0$ as $\varepsilon \downarrow 0$.
Thus, the first three terms in the asymptotic expansion of f are

$$f(\varepsilon) \sim 1 - \varepsilon + \varepsilon^2 \qquad \text{as } \varepsilon \downarrow 0.$$

Observation:

In the previous example, the exponential term $e^{-1/\varepsilon}$ is negligible compared to any power of ε as $\varepsilon \downarrow 0$. This is because $e^{-1/\varepsilon} = o(\varepsilon^\alpha)$ for all values of α as $\varepsilon \downarrow 0$.
This example shows that **two functions can have the same asymptotic expansion.**

2.2 Notes about accuracy and convergence of an asymptotic series

It is common to expect that to improve the accuracy of an asymptotic expansion, we need to include more terms in the expansion (like with a convergent Taylor series). However, this is not always the case. In fact, it can happen that including more terms in the expansion actually makes the approximation worse. This is because **an asymptotic expansion is not necessarily a convergent series**, and even if it is convergent, it may converge to a different function than the one we are trying to approximate.

An asymptotic expansion only makes a statement about the series in the limit of $\varepsilon \downarrow \varepsilon_0$, whereas increasing the number of terms is saying something about the series as $n \to \infty$.

Importantly, asymptotic approximations are most valuable when they can give **insights** into the structure of a solution, rather than when they are used to obtaining **numerical accuracy**, which in this case, numerical methods are worth considering.

3 Lecture 3: Manipulating Asymptotic Expansions

3.1 Operations on Asymptotic Expansions

- Two asymptotic expansions may be added term by term, provided they are expressed with respect to the same asymptotic sequence. Note that, after addition, the resulting series may need to be re-ordered if leading terms cancel.

- Two asymptotic expansions may be multiplied, provided they are expressed with respect to a common asymptotic sequence that is totally ordered by asymptotic dominance and whose products can be re-expressed within the same ordered scale.

Observation:

Differentiation: Suppose that

$$f(x,\varepsilon) \sim \phi_1(x,\varepsilon) + \phi_2(x,\varepsilon) \quad \text{as } \varepsilon \downarrow 0.$$

What can we say about the asymptotic expansion of $\frac{\partial f}{\partial x}(x,\varepsilon)$ as $\varepsilon \downarrow 0$?
In general, termwise differentiation is not justified without additional assumptions. In

particular, it is not necessarily true that

$$\frac{\partial f}{\partial x}(x,\varepsilon) \sim \frac{\partial \phi_1}{\partial x}(x,\varepsilon) + \frac{\partial \phi_2}{\partial x}(x,\varepsilon) \quad \text{as } \varepsilon \downarrow 0.$$

Differentiation does not necessarily preserve the asymptotic ordering of terms in an expansion. For example, letting

$$\phi_1(x,\varepsilon) = 1 + x, \qquad \phi_2(x,\varepsilon) = \varepsilon \sin(x/\varepsilon),$$

for $0 \le x \le 1$ and $\varepsilon > 0$, we have

$$\phi_2(x,\varepsilon) = o(\phi_1(x,\varepsilon)) \quad \text{as } \varepsilon \downarrow 0,$$

but

$$\frac{\partial \phi_2}{\partial x}(x,\varepsilon) = \cos(x/\varepsilon)$$

is not $o\big(\frac{\partial \phi_1}{\partial x}(x,\varepsilon)\big)$ as $\varepsilon \downarrow 0$.

When is termwise differentiation valid?:

Termwise differentiation of an asymptotic expansion is justified under suitable structural conditions. Common situations include:

- **Taylor (power series) expansions:** If a function admits a Taylor expansion with a remainder estimate valid in a neighbourhood of the expansion point, then termwise differentiation is valid.
- **Algebraic (power–logarithmic) scales:** For scales such as $\varepsilon^{\alpha}(\log \varepsilon)^{\beta}$, the asymptotic ordering is preserved under differentiation (after possible reindexing).
- **Exponential scales:** For scales of the form $e^{-k/\varepsilon}$ $(k > 0)$, differentiation introduces only algebraic prefactors and does not alter the exponential dominance hierarchy.

Failure Case: Differentiation may fail when the remainder contains rapid oscillations (for example, terms such as $\varepsilon \sin(x/\varepsilon)$), since a function may be small while its derivative is $O(1)$.

Differentiation of an Asymptotic Expansion:

Let $f(x,\varepsilon)$ admit an asymptotic expansion as $\varepsilon \to \varepsilon_0$ with respect to the asymptotic sequence $\{\phi_k(\varepsilon)\}$:

$$f(x,\varepsilon) \sim \sum_{k=1}^{\infty} a_k(x)\, \phi_k(\varepsilon).$$

Assume that each coefficient $a_k(x)$ is differentiable with respect to x, and that the partial derivative $f_x(x,\varepsilon)$ also admits an asymptotic expansion as $\varepsilon \to \varepsilon_0$ with respect to the same scale functions $\{\phi_k(\varepsilon)\}$.

Then

$$\frac{\partial f}{\partial x}(x,\varepsilon) \sim \sum_{k=1}^{\infty} a_k'(x)\,\phi_k(\varepsilon), \qquad \varepsilon \to \varepsilon_0.$$

Application of the differentiation theorem:

Let

$$f(x,\varepsilon) = \frac{1}{1+\varepsilon x}, \qquad x \in [0,M], \qquad \varepsilon \downarrow 0.$$

Step 1: Asymptotic expansion of f.
For $|\varepsilon x| < 1$ we have the geometric series expansion

$$f(x,\varepsilon) = \sum_{k=0}^{\infty} (-1)^k x^k \varepsilon^k.$$

Thus,

$$f(x,\varepsilon) \sim \sum_{k=0}^{\infty} a_k(x)\varepsilon^k, \quad a_k(x) = (-1)^k x^k.$$

Step 2: Expansion of the derivative.
Differentiating the closed-form expression gives

$$f_x(x,\varepsilon) = -\frac{\varepsilon}{(1+\varepsilon x)^2}.$$

Using the binomial expansion for $(1+\varepsilon x)^{-2}$,

$$(1+\varepsilon x)^{-2} = \sum_{k=0}^{\infty} (k+1)(-1)^k x^k \varepsilon^k,$$

we obtain

$$f_x(x,\varepsilon) \sim \sum_{k=1}^{\infty} k(-1)^k x^{k-1} \varepsilon^k.$$

Step 3: Comparison of coefficients.
Since

$$a_k'(x) = k(-1)^k x^{k-1},$$

we conclude that

$$f_x(x,\varepsilon) \sim \sum_{k=0}^{\infty} a_k'(x)\varepsilon^k.$$

Thus, the differentiation theorem applies.

Differentiation of an asymptotic power series (Stronger version):

Let

$$f(x,\varepsilon) = \frac{1}{1+\varepsilon x}, \qquad x \in [0,M], \qquad \varepsilon \downarrow 0,$$

where $M > 0$ is fixed.

Step 1: Expansion using the geometric series.
Recall the finite geometric series identity

$$1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}.$$

Rearranging,

$$\frac{1}{1-r} = \sum_{k=0}^{n} r^k + \frac{r^{n+1}}{1-r}.$$

We apply this with $r = -\varepsilon x$. Then, provided $|\varepsilon x| < 1$, we obtain the exact identity

$$\frac{1}{1+\varepsilon x} = \sum_{k=0}^{n} (-\varepsilon x)^k + \frac{(-\varepsilon x)^{n+1}}{1+\varepsilon x}.$$

Since $x \in [0, M]$, we have

$$|\varepsilon x| \leq \varepsilon M.$$

Thus, if $\varepsilon < 1/M$, then $|\varepsilon x| < 1$ for all $x \in [0, M]$, and the above formula holds uniformly on this interval.

Step 2: Identifying the remainder.
The remainder term is therefore known explicitly:

$$R_n(x, \varepsilon) = \frac{(-\varepsilon x)^{n+1}}{1+\varepsilon x}.$$

For $x \in [0, M]$,

$$|R_n(x, \varepsilon)| \leq \frac{(\varepsilon M)^{n+1}}{1} = M^{n+1}\varepsilon^{n+1}.$$

Hence,

$$R_n(x, \varepsilon) = O(\varepsilon^{n+1}) \quad \text{uniformly for } x \in [0, M].$$

Step 3: Differentiating the remainder.
Since we know R_n explicitly, we may differentiate it directly. Every term in $\partial_x R_n$ contains at least one factor ε^{n+1}, and therefore

$$\partial_x R_n(x, \varepsilon) = O(\varepsilon^{n+1}) \quad \text{uniformly for } x \in [0, M].$$

Thus, differentiation preserves the asymptotic order.

Step 4: Direct differentiation of f.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(1 + \varepsilon x)^{-1} = -\frac{\varepsilon}{(1+\varepsilon x)^2}.$$

Step 5: Term-by-term differentiation.
Differentiating the expansion term by term gives

$$\frac{\partial f}{\partial x} \sim -\varepsilon + 2x\varepsilon^2 - 3x^2\varepsilon^3 + \cdots \qquad (\varepsilon \downarrow 0).$$

Verification.
Using the binomial expansion

$$(1+\varepsilon x)^{-2} = 1 - 2\varepsilon x + 3(\varepsilon x)^2 - \cdots,$$

we obtain

$$-\varepsilon(1+\varepsilon x)^{-2} = -\varepsilon + 2x\varepsilon^2 - 3x^2\varepsilon^3 + \cdots,$$

which agrees with the term-by-term differentiation.

Integration of an Asymptotic Expansion:

Assume that as $\varepsilon \downarrow 0$,

$$f(x,\varepsilon) = a_1(x)\phi_1(\varepsilon) + a_2(x)\phi_2(\varepsilon) + R(x,\varepsilon),$$

where $a_1, a_2 \in L^1([a,b])$ and the remainder satisfies

$$R(x,\varepsilon) = o(\phi_2(\varepsilon)) \quad \text{uniformly for } x \in [a,b].$$

Then

$$\int_a^b f(x,\varepsilon)\,dx \sim \left(\int_a^b a_1(x)\,dx\right)\phi_1(\varepsilon) + \left(\int_a^b a_2(x)\,dx\right)\phi_2(\varepsilon) \quad \text{as } \varepsilon \downarrow 0.$$

In particular, the interval $[a,b]$ must lie in a region of x where the expansion holds uniformly.

Uniform smallness:

We say that $R(x,\varepsilon) = o(\phi(\varepsilon))$ uniformly for $x \in [a,b]$ if

$$\sup_{x\in[a,b]} \left|\frac{R(x,\varepsilon)}{\phi(\varepsilon)}\right| \to 0 \quad \text{as } \varepsilon \to 0.$$

Example:

Find the first two terms in the expansion of

$$f(\varepsilon) = \int_0^1 e^{\varepsilon x^2}\,dx \qquad (\varepsilon \to 0).$$

Solution. For each fixed x,

$$e^{\varepsilon x^2} = 1 + \varepsilon x^2 + O(\varepsilon^2).$$

Since $0 \le x \le 1$, the remainder estimate is uniform in x. Hence, we may integrate term by term:

$$f(\varepsilon) = \int_0^1 1\,dx + \varepsilon\int_0^1 x^2\,dx + O(\varepsilon^2).$$

Evaluating the integrals,

$$f(\varepsilon) = 1 + \frac{1}{3}\varepsilon + O(\varepsilon^2).$$

Example:

Find the first two terms in the expansion of

$$f(\varepsilon) = \int_0^1 \frac{dx}{\varepsilon^2 + x^2} \qquad \text{as } \varepsilon \downarrow 0.$$

Solution. Assume $\varepsilon > 0$. We can evaluate the integral exactly:

$$\int \frac{dx}{\varepsilon^2 + x^2} = \frac{1}{\varepsilon} \arctan\left(\frac{x}{\varepsilon}\right),$$

hence

$$f(\varepsilon) = \frac{1}{\varepsilon} \arctan\left(\frac{x}{\varepsilon}\right)\Big|_0^1 = \frac{1}{\varepsilon} \arctan\left(\frac{1}{\varepsilon}\right).$$

Now expand as $\varepsilon \downarrow 0$. Using

$$\arctan\left(\frac{1}{\varepsilon}\right) = \frac{\pi}{2} - \arctan(\varepsilon),$$

and the Taylor expansion

$$\arctan(\varepsilon) = \varepsilon - \frac{\varepsilon^3}{3} + O(\varepsilon^5),$$

we obtain

$$\arctan\left(\frac{1}{\varepsilon}\right) = \frac{\pi}{2} - \varepsilon + \frac{\varepsilon^3}{3} + O(\varepsilon^5).$$

Therefore,

$$f(\varepsilon) = \frac{1}{\varepsilon}\left(\frac{\pi}{2} - \varepsilon + \frac{\varepsilon^3}{3} + O(\varepsilon^5)\right) = \frac{\pi}{2\varepsilon} - 1 + \frac{\varepsilon^2}{3} + O(\varepsilon^4).$$

In particular, the first two terms are

$$\boxed{f(\varepsilon) \sim \frac{\pi}{2\varepsilon} - 1 \qquad (\varepsilon \downarrow 0).}$$

Inner/outer expansion (logarithmic singularity):

Find the first two terms in the expansion of

$$f(\varepsilon) = \int_0^{\pi/3} \frac{dx}{\varepsilon^2 + \sin x}, \qquad \varepsilon \downarrow 0.$$

Idea. The integrand is well-behaved for $x > 0$, but $\sin x \sim x$ as $x \to 0$, so the integral has a *logarithmic* contribution from a thin region near $x = 0$. We split the interval at an intermediate point $\delta = \delta(\varepsilon)$ such that

$$\varepsilon^2 \ll \delta \ll 1,$$

and write

$$f(\varepsilon) = \int_0^{\delta} \frac{dx}{\varepsilon^2 + \sin x} + \int_{\delta}^{\pi/3} \frac{dx}{\varepsilon^2 + \sin x} =: I_{\text{in}} + I_{\text{out}}.$$

Inner region ($0 \le x \le \delta$). Here $x \ll 1$, so $\sin x = x + O(x^3)$. Introduce the inner scaling

$$x = \varepsilon^2 X, \qquad dx = \varepsilon^2 dX,$$

so that, since $\varepsilon^2 X \le \delta \ll 1$,

$$\sin(\varepsilon^2 X) = \varepsilon^2 X + O(\varepsilon^6 X^3).$$

Then

$$I_{\text{in}} = \int_0^{\delta/\varepsilon^2} \frac{\varepsilon^2\, dX}{\varepsilon^2 + \sin(\varepsilon^2 X)} = \int_0^{\delta/\varepsilon^2} \frac{dX}{1 + X + O(\varepsilon^4 X^3)}.$$

To obtain the first two terms in $f(\varepsilon)$ (a logarithm plus a constant), it is enough to keep the leading inner approximation:

$$I_{\text{in}} = \int_0^{\delta/\varepsilon^2} \frac{dX}{1+X} + o(1) = \log\Bigl(1 + \frac{\delta}{\varepsilon^2}\Bigr) + o(1) = -\,2\log\varepsilon + \log\delta + o(1).$$

Outer region ($\delta \le x \le \pi/3$). Here $\sin x$ is bounded away from 0 (since $x \ge \delta > 0$), so we may expand

$$\frac{1}{\varepsilon^2 + \sin x} = \frac{1}{\sin x}\,\frac{1}{1 + \varepsilon^2/\sin x} = \frac{1}{\sin x} + O(\varepsilon^2) \quad \text{uniformly for } x \in [\delta, \pi/3].$$

Hence,

$$I_{\text{out}} = \int_{\delta}^{\pi/3} \frac{dx}{\sin x} + o(1).$$

Using $\int \csc x\, dx = \log\tan(x/2) + C$, we find

$$I_{\text{out}} = \Bigl[\log\tan(x/2)\Bigr]_{\delta}^{\pi/3} + o(1) = \log\tan\Bigl(\frac{\pi}{6}\Bigr) - \log\tan\Bigl(\frac{\delta}{2}\Bigr) + o(1).$$

Since $\tan(\pi/6) = 1/\sqrt{3}$ and $\tan(\delta/2) \sim \delta/2$ as $\delta \to 0$,

$$I_{\text{out}} = \log\Bigl(\frac{1}{\sqrt{3}}\Bigr) - \log\Bigl(\frac{\delta}{2}\Bigr) + o(1) = -\log\delta + \log\Bigl(\frac{2}{\sqrt{3}}\Bigr) + o(1).$$

Matching and final expansion. Adding I_{in} and I_{out}, the $\log\delta$ terms cancel:

$$f(\varepsilon) = \bigl(-2\log\varepsilon + \log\delta\bigr) + \bigl(-\log\delta + \log(2/\sqrt{3})\bigr) + o(1).$$

Therefore,

$$\boxed{f(\varepsilon) = -\,2\log\varepsilon + \log\Bigl(\frac{2}{\sqrt{3}}\Bigr) + o(1), \qquad \varepsilon \downarrow 0.}$$

4 Lecture 4: Asymptotic Solution of Algebraic Equations

4.1 Uniformity of Asymptotic Approximations and Expansions

In some cases, other parameters or variables that a function depends on can interfere with the accuracy, or even existence, of an asymptotic expansion. For example, if we have a function $f(x,\varepsilon)$ that depends on both x and ε, then the asymptotic approximation of f as $\varepsilon \downarrow 0$ may not be valid for all values of x. In particular, the approximation may only be valid for x in a certain range, and it may break down as x approaches the endpoints of that range.

Example (Failure of uniformity as $x \to 1$):

Consider the algebraic equation for $y = y(x)$,

$$(y-1)(y-x) = \varepsilon y, \qquad \varepsilon \downarrow 0,$$

and restrict to $x \in (0,1)$.

Step 1: Reduced problem ($\varepsilon = 0$). Setting $\varepsilon = 0$ gives

$$(y-1)(y-x) = 0,$$

so the reduced equation has two solution branches,

$$y = 1 \qquad \text{and} \qquad y = x.$$

We construct asymptotic expansions for the two branches.

Step 2: Expansion near the branch $y = 1$ (for fixed $x \in (0,1)$). Assume

$$y \sim 1 + \varepsilon y_1(x) + \cdots .$$

Then

$$y - 1 \sim \varepsilon y_1, \qquad y - x \sim (1-x) + \varepsilon y_1, \qquad \varepsilon y \sim \varepsilon(1) + O(\varepsilon^2).$$

Substituting into $(y-1)(y-x) = \varepsilon y$ and keeping the leading $O(\varepsilon)$ terms gives

$$(\varepsilon y_1)(1-x) = \varepsilon,$$

so

$$y_1(x) = \frac{1}{1-x}.$$

Hence, for fixed $x \in (0,1)$,

$$\boxed{y(x) \sim 1 + \frac{\varepsilon}{1-x} \qquad (\varepsilon \downarrow 0).}$$

Step 3: Why this approximation is not uniformly valid on $(0,1)$. Uniform validity on an interval means the relative error can be made small with a single ε_2

that works for *all* x in the interval. The approximation above fails to be uniform on $(0,1)$ because the correction term

$$\frac{\varepsilon}{1-x}$$

blows up as $x \to 1^-$. In particular, if x approaches 1 so that $1-x$ is comparable to ε, say $1-x = O(\varepsilon)$, then

$$\frac{\varepsilon}{1-x} = O(1),$$

so the "small correction" is no longer small. Thus, the expansion about $y = 1$ breaks down as $x \to 1^-$.
Equivalently, to ensure the correction remains small we must require

$$\frac{\varepsilon}{1-x} \ll 1, \qquad \text{i.e.} \qquad 1-x \gg \varepsilon.$$

Therefore, the approximation

$$y(x) \sim 1 + \frac{\varepsilon}{1-x}$$

is valid only away from $x = 1$, for example for $x \leq 1-\delta$ with fixed $\delta > 0$ (independent of ε), but it is *not* uniformly valid on the full interval $(0,1)$.

Uniformly Valid Approximation:

An asymptotic approximation

$$f(x,\varepsilon) \sim \phi(x,\varepsilon) \qquad (\varepsilon \downarrow 0)$$

is said to be **uniformly valid** for $x \in I$ if, given any positive constant δ, we can find an ε_2 (independent of x and ε) such that, for $0 < \varepsilon < \varepsilon_2$,

$$|f(x,\varepsilon) - \phi(x,\varepsilon)| \leq \delta|\phi(x,\varepsilon)| \qquad \text{for all } x \in I.$$

Criteria for Uniformity:

Assume $f(x,\varepsilon)$, $\phi(x,\varepsilon)$, and $\phi_0(\varepsilon)$ are continuous for $a \leq x \leq b$ and $0 < \varepsilon < \varepsilon_1$.

(a) If $f(x,\varepsilon) \sim \phi(x,\varepsilon)$ for $a \leq x \leq b$, and if $|\phi(x,\varepsilon)|$ is monotonically decreasing in ε, then the asymptotic approximation is uniformly valid for $a \leq x \leq b$.

(b) Suppose $f(x,\varepsilon) \sim \phi(x,\varepsilon)$ for $a < x < b$ and $f(x_0,\varepsilon) \sim \phi_0(\varepsilon)$ for $x_0 = a$ or $x_0 = b$. If $\phi_0(\varepsilon)$ is nonzero and

$$\lim_{\varepsilon\to 0} \frac{\phi(x_0,\varepsilon)}{\phi_0(\varepsilon)} \neq 1,$$

then $f \sim \phi$ is **not** uniformly valid for $a < x < b$.

4.2 Uniformly Valid Asymptotic Expansions

Suppose
$$f(x,\varepsilon) \sim a_1(x)\phi_1(x,\varepsilon) + a_2(x)\phi_2(x,\varepsilon) \qquad (\varepsilon \downarrow 0).$$

The expansion is **uniformly valid** for $x \in I$ if, given any positive constant δ, we can find an ε_2 (independent of x and ε) such that, for $0 < \varepsilon < \varepsilon_2$ and all $x \in I$,

$$|f(x,\varepsilon) - a_1(x)\phi_1(x,\varepsilon)| \leq \delta|a_1(x)\phi_1(x,\varepsilon)|,$$
$$|f(x,\varepsilon) - a_1(x)\phi_1(x,\varepsilon) - a_2(x)\phi_2(x,\varepsilon)| \leq \delta|a_2(x)\phi_2(x,\varepsilon)|.$$

This definition extends in a straightforward manner to expansions involving more scale functions.

Examples

A simple Taylor expansion:

Let
$$f(x,\varepsilon) = \sin(x+\varepsilon).$$
We investigate whether the approximation
$$f(x,\varepsilon) \sim \sin x \qquad (\varepsilon \downarrow 0)$$
is uniformly valid on various intervals in x.

Step 1: Expand using Taylor's theorem.
Expanding about $\varepsilon = 0$ gives
$$\sin(x+\varepsilon) = \sin x + \varepsilon\cos(x+\xi),$$
for some ξ between 0 and ε.
Hence,
$$|f(x,\varepsilon) - \sin x| = |\varepsilon\cos(x+\xi)| \leq |\varepsilon|.$$

Step 2: Uniform validity on $\frac{\pi}{4} \leq x \leq \frac{3\pi}{4}$.
On this interval, $\sin x$ is bounded away from zero. Therefore,
$$\sup_{x\in[\pi/4,\,3\pi/4]} |f(x,\varepsilon) - \sin x| \leq |\varepsilon| \to 0.$$
Thus, the approximation is uniformly valid on this interval.

Step 3: Why uniformity fails near $x = 0$.
Now consider the interval $0 < x < \pi/2$.
At $x = 0$,
$$f(0,\varepsilon) = \sin(\varepsilon) \sim \varepsilon, \qquad \phi(0,\varepsilon) = \sin(0) = 0.$$
Thus, the leading term $\sin x$ vanishes at $x = 0$, while the true solution is $O(\varepsilon)$. The relative error therefore does not remain small uniformly.
The difficulty arises because the leading term becomes small at the endpoint. Hence, the approximation is not uniform on $0 < x < \pi/2$.

A 'boundary layer' example:

Consider
$$\varepsilon y'' + 2y' + 2y = 0, \qquad 0 < x < 1.$$
The exact solution is
$$y(x) = e^{\alpha x/\varepsilon} + e^{\beta x/\varepsilon},$$
where
$$\alpha = -1 + \sqrt{1 - 2\varepsilon}, \qquad \beta = -1 - \sqrt{1 - 2\varepsilon}.$$

Step 1: Expand the exponents for small ε.
For small ε,
$$\alpha \sim -\varepsilon - \frac{\varepsilon^2}{2}, \qquad \beta \sim -2 + \varepsilon.$$
Hence,
$$y(x) \sim e^{-x} + e^{-2x/\varepsilon}.$$

Step 2: Outer (nonuniform) approximation.
For fixed $x > 0$, the term $e^{-2x/\varepsilon}$ is exponentially small. Dropping it gives
$$y(x) \sim e^{-x}.$$
This is an excellent approximation for fixed $x > 0$.

Step 3: Where does it fail?
If $x = O(\varepsilon)$, say $x = \varepsilon\xi$, then
$$e^{-2x/\varepsilon} = e^{-2\xi},$$
which is $O(1)$.
Thus near $x = 0$ the neglected exponential is of the same order as the retained term.
The approximation therefore breaks down in a thin region
$$x = O(\varepsilon),$$
called a *boundary layer*.
Hence, $y(x) \sim e^{-x}$ is not uniformly valid on $0 < x < 1$.
If instead we retain both exponentials,
$$y(x) \sim e^{-x} + e^{-2x/\varepsilon},$$
we obtain a uniformly valid approximation.

Loss of uniformity on an unbounded interval:

Consider
$$y'' + 2\varepsilon y' + y = 0, \qquad 0 < t.$$
The exact solution is
$$y(t) = e^{-\varepsilon t} \sin\left(t\sqrt{1 - \varepsilon^2}\right).$$

Step 1: Expand for small ε.

We use
$$\sqrt{1-\varepsilon^2} \sim 1 - \frac{\varepsilon^2}{2}, \qquad e^{-\varepsilon t} \sim 1 - \varepsilon t.$$
Thus,
$$y(t) \sim (1-\varepsilon t)\sin t = \sin t - \varepsilon t \sin t.$$

Step 2: Why is this not uniform for $0 < t < \infty$?
As t increases, the correction term
$$\varepsilon t \sin t$$
eventually becomes as large as $\sin t$.
Indeed, when $t = O(1/\varepsilon)$, the correction is $O(1)$.
Thus, the expansion is not well-ordered on the whole half-line. The second term eventually dominates the first.

Step 3: Uniform validity on bounded intervals.
However, if $0 \le t \le T$ for fixed T, then
$$|\varepsilon t| \le \varepsilon T \to 0,$$
and the expansion remains well-ordered.
Hence, the approximation is uniform on any bounded interval, but not on $0 < t < \infty$.

4.3 Regular Perturbation Problems

Regular perturbation problem:

Consider a problem depending on a small parameter ε. If, after setting $\varepsilon = 0$, the reduced problem
$$F(x, 0) = 0$$
has a solution that persists for sufficiently small ε and the solution $x(\varepsilon)$ depends smoothly on ε (with no loss of conditions or change in the structure of the problem), then the problem is called a *regular perturbation problem.*

To illustrate, we start with a simple example.

Example (First solve then expand):

Solve
$$x^2 + 2\varepsilon x - 1 = 0, \qquad \varepsilon \ll 1.$$

Solution. We solve the quadratic exactly and then expand for small ε.
Using the quadratic formula,
$$x = \frac{-2\varepsilon \pm \sqrt{(2\varepsilon)^2 - 4(1)(-1)}}{2} = \frac{-2\varepsilon \pm \sqrt{4\varepsilon^2 + 4}}{2}.$$
Factor out 4 inside the square root:
$$x = \frac{-2\varepsilon \pm 2\sqrt{1+\varepsilon^2}}{2} = -\varepsilon \pm \sqrt{1+\varepsilon^2}.$$

Now expand $\sqrt{1+\varepsilon^2}$ for small ε. Using the binomial expansion

$$\sqrt{1+u} = 1 + \frac{u}{2} - \frac{u^2}{8} + O(u^3),$$

with $u = \varepsilon^2$, we obtain

$$\sqrt{1+\varepsilon^2} = 1 + \frac{\varepsilon^2}{2} - \frac{\varepsilon^4}{8} + O(\varepsilon^6).$$

Substituting back:

First root:

$$\begin{aligned} x_+ &= -\varepsilon + \left(1 + \frac{\varepsilon^2}{2} - \frac{\varepsilon^4}{8} + O(\varepsilon^6)\right) \\ &= 1 - \varepsilon + \frac{\varepsilon^2}{2} - \frac{\varepsilon^4}{8} + O(\varepsilon^6). \end{aligned}$$

Second root:

$$\begin{aligned} x_- &= -\varepsilon - \left(1 + \frac{\varepsilon^2}{2} - \frac{\varepsilon^4}{8} + O(\varepsilon^6)\right) \\ &= -1 - \varepsilon - \frac{\varepsilon^2}{2} + \frac{\varepsilon^4}{8} + O(\varepsilon^6). \end{aligned}$$

Therefore, the first two terms in the asymptotic expansions are

$$\boxed{x_+ \sim 1 - \varepsilon, \qquad x_- \sim -1 - \varepsilon \qquad (\varepsilon \to 0).}$$

If one more term is retained,

$$x_+ \sim 1 - \varepsilon + \frac{\varepsilon^2}{2}, \qquad x_- \sim -1 - \varepsilon - \frac{\varepsilon^2}{2}.$$

Example (expand first, then solve):

Solve

$$x^2 + 2\varepsilon x - 1 = 0, \qquad \varepsilon \ll 1.$$

Solution.

We first assess how many solutions there are and their approximate location. Rewriting the equation as

$$1 - x^2 = 2\varepsilon x,$$

we observe that for $\varepsilon = 0$ the solutions are $x = \pm 1$. For small $\varepsilon > 0$, the right-hand side is small, so we expect two real solutions: one near $x = 1$ and one near $x = -1$. This immediately tells us what the expansion should *not* look like. It cannot be of the form $x \sim \varepsilon x_0 + \cdots$, since that would imply $x \to 0$ as $\varepsilon \to 0$, which is false. Similarly, it cannot be of the form $x \sim \varepsilon^{-1} x_0 + \cdots$, since the solutions remain bounded as $\varepsilon \to 0$. Therefore, we seek an expansion of the form

$$x \sim x_0 + \varepsilon^{\alpha} x_1 + \cdots, \qquad \alpha > 0,$$

where α is initially unknown. Allowing α to vary gives flexibility and lets the equation determine the correct scaling.

Substitute this ansatz (educated guess) into the equation:

$$\left(x_0 + \varepsilon^\alpha x_1 + \cdots\right)^2 + 2\varepsilon\left(x_0 + \varepsilon^\alpha x_1 + \cdots\right) - 1 = 0.$$

Expanding,

$$x_0^2 + 2\varepsilon^\alpha x_0 x_1 + \cdots + 2\varepsilon x_0 + 2\varepsilon^{1+\alpha} x_1 + \cdots - 1 = 0.$$

Now group terms according to powers of ε.

Order $O(1)$.
As $\varepsilon \downarrow 0$, the equation must still hold, so the $O(1)$ terms give

$$x_0^2 - 1 = 0.$$

Hence,

$$x_0 = \pm 1.$$

This confirms our earlier observation that the solutions are near ± 1.

Determining α (balancing argument).
At the next level, we observe that there is a term

$$2\varepsilon x_0,$$

which is $O(\varepsilon)$. For the equation to hold, this term must be balanced by another term of the same order.
The only possible candidate is

$$2\varepsilon^\alpha x_0 x_1.$$

Therefore, in order for these two contributions to be of the same order, we must have

$$\varepsilon^\alpha \sim \varepsilon,$$

which implies

$$\alpha = 1.$$

Thus, the correct expansion is

$$x \sim x_0 + \varepsilon x_1 + \cdots .$$

Order $O(\varepsilon)$.
Collecting the $O(\varepsilon)$ terms gives

$$2x_0 x_1 + 2x_0 = 0.$$

Factor out $2x_0$:

$$2x_0(x_1 + 1) = 0.$$

Since $x_0 \neq 0$, we obtain

$$x_1 = -1.$$

Conclusion.
Therefore, the two solutions admit the first-order approximations

$$x \sim \pm 1 - \varepsilon \qquad (\varepsilon \to 0).$$

The process may be continued to determine higher-order terms if desired.

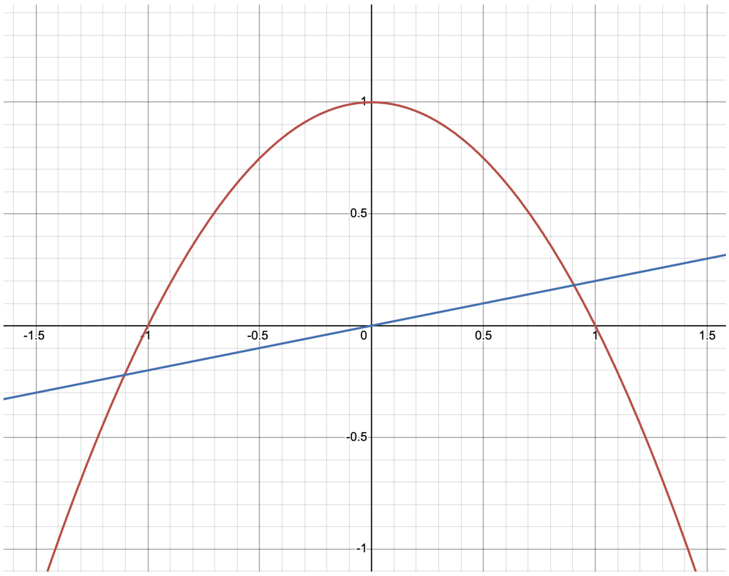


Figure 4: Graph of the functions $f(x, \varepsilon) = 1 - x^2$ and $g(x, \varepsilon) = 2\varepsilon x$. For small ε, the roots are close to ± 1. As ε increases, the roots move away from ± 1, but for small ε they remain close to these values, illustrating the regular perturbation structure.

Note about the notation:

In the derivation above, we referred to the equation satisfied by x_0 as the $O(1)$ *equation*. Strictly speaking, this terminology is imprecise.
According to the definition of Big–O notation, a function $f(\varepsilon)$ is $O(1)$ as $\varepsilon \to 0$ if it remains bounded. With this definition, *every term* in the equation

$$x^2 + 2\varepsilon x - 1 = 0$$

is $O(1)$ as $\varepsilon \to 0$, including the term $2\varepsilon x$, since $\varepsilon x \to 0$ and is therefore bounded.
What we actually mean by the "$O(1)$ equation" is the equation formed by collecting the *leading-order terms* — that is, the terms that do not vanish as $\varepsilon \to 0$. In this example, those are the terms independent of ε, namely

$$x_0^2 - 1 = 0.$$

If one wished to be completely precise, one could introduce a refined notation, for example writing

$$f(\varepsilon) = O_s(1)$$

to mean that $f(\varepsilon)$ is bounded but not $o(1)$; that is, $f = O(1)$ and $f \not\to 0$. However, such notation is rarely used in practice.
Instead, it is standard in perturbation theory to use the phrase "$O(1)$ terms" informally to mean *leading-order terms* in the asymptotic expansion.
Thus, the symbol $O(\cdot)$ is used in two related but distinct ways:

- as a precise asymptotic bound (the definition of Big–O),
- as a bookkeeping device to identify the relative size of terms in a perturbation expansion.

The intended meaning should always be clear from context.

5 Lecture 5: Asymptotic Expansions (Part II)

5.1 Asymptotic solution of algebraic equations (continued)

5.1.1 Singular Perturbation Problems

Example (Singular perturbation problem):

Consider the quadratic equation

$$\varepsilon x^2 + 2x - 1 = 0, \qquad \varepsilon \ll 1.$$

Observe that if we set $\varepsilon = 0$, the reduced equation is linear, that is, the order of the equation is reduced. This is important because it fundamentally alters the nature of the equation and can give rise to what is known as a *singular perturbation problem*.
If we approach the problem in the same way as in the regular perturbation case, we would seek an expansion of the form

$$x \sim x_0 + \varepsilon^\alpha x_1 + \cdots, \qquad \alpha > 0,$$

and substitute into the equation. The leading-order balance gives

$$2x_0 - 1 = 0,$$

which implies $x_0 = \frac{1}{2}$. The order $O(\varepsilon)$ terms give

$$2x_1 + x_0^2 = 0,$$

which implies $x_1 = -\frac{1}{8}$. Thus, we obtain the expansion

$$x \sim \frac{1}{2} - \frac{1}{8}\varepsilon + \cdots.$$

However, this expansion only captures one of the two roots of the equation.
Note that the problem is **singular** in the sense that if we set $\varepsilon = 0$, the nature of the equation changes (from quadratic to linear), and one root is lost.
To capture the missing root, we assume a more general scaling

$$x \sim \varepsilon^\gamma \left(x_0 + \varepsilon^\alpha x_1 + \cdots\right),$$

where $\alpha > 0$ so the expansion is well-ordered, and γ is an unknown exponent to be determined.
Substituting into the equation, we examine the order of each term:

$$\varepsilon x^2 \sim \varepsilon^{1+2\gamma}, \qquad 2x \sim \varepsilon^\gamma, \qquad -1 \sim \varepsilon^0.$$

For the equation to balance as $\varepsilon \to 0$, at least two of the dominant terms must be of the same order.
There are three possible balances:

1. Balance $2x$ and -1:

   $$\gamma = 0.$$

   This gives the regular $O(1)$ solution already obtained.

2. Balance εx^2 and -1:
$$1 + 2\gamma = 0 \quad \Rightarrow \quad \gamma = -\tfrac{1}{2}.$$
In this case, $2x \sim \varepsilon^{-1/2}$, which dominates the other terms. Hence, this balance is inconsistent and must be discarded.

3. Balance εx^2 and $2x$:
$$1 + 2\gamma = \gamma \quad \Rightarrow \quad \gamma = -1.$$
Then
$$\varepsilon x^2 \sim \varepsilon^{-1}, \qquad 2x \sim \varepsilon^{-1}, \qquad -1 \sim \varepsilon^0.$$
Thus, the first two terms dominate and balance each other, while the constant term is of lower order. This balance is consistent.

Hence, the second root scales like $x = O(\varepsilon^{-1})$. Writing
$$x = \frac{X}{\varepsilon},$$
and substituting into the equation gives
$$X^2 + 2X - \varepsilon = 0.$$
We now seek a regular expansion
$$X \sim X_0 + \varepsilon X_1 + \cdots .$$
Substituting gives, at leading order,
$$X_0^2 + 2X_0 = 0,$$
so
$$X_0(X_0 + 2) = 0.$$
The nontrivial solution is $X_0 = -2$, corresponding to the large root. Thus,
$$x \sim -\frac{2}{\varepsilon} + \cdots .$$
Proceeding to the next order yields
$$x \sim -\frac{2}{\varepsilon} - \frac{1}{2} + O(\varepsilon).$$
We therefore obtain the two asymptotic solutions
$$x_1 \sim \frac{1}{2} - \frac{1}{8}\varepsilon + \cdots , \qquad x_2 \sim -\frac{2}{\varepsilon} - \frac{1}{2} + \cdots .$$
This example illustrates how balancing the dominant powers of ε reveals multiple solution scalings in a singular perturbation problem.

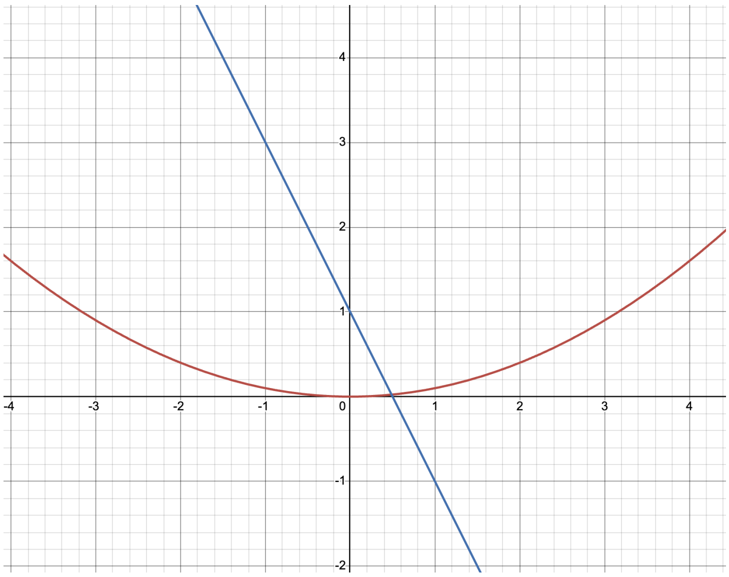


Figure 5: Graph of the functions $f(x,\varepsilon) = \varepsilon x^2$ and $g(x,\varepsilon) = 1 - 2x$. For small ε, the roots are close to $1/2$ and $-\infty$.

5.2 Asymptotic solution of transcendental equations

Transcendental equation: $x^2 + e^{\varepsilon x} = 5$ as $\varepsilon \to 0$:

Consider

$$x^2 + e^{\varepsilon x} = 5, \qquad \varepsilon \ll 1.$$

We seek asymptotic expansions of the real solutions as $\varepsilon \to 0$.

Step 1: Leading-order problem. Since $\varepsilon \to 0$ and we expect $x = O(1)$, we expand

$$e^{\varepsilon x} = 1 + \varepsilon x + \frac{\varepsilon^2 x^2}{2} + O(\varepsilon^3).$$

Substituting into the equation gives

$$x^2 + 1 + \varepsilon x + \frac{\varepsilon^2 x^2}{2} + O(\varepsilon^3) = 5,$$

or

$$x^2 - 4 + \varepsilon x + \frac{\varepsilon^2 x^2}{2} + O(\varepsilon^3) = 0.$$

At leading order we obtain

$$x_0^2 - 4 = 0 \quad \Rightarrow \quad x_0 = \pm 2.$$

Thus, there are two $O(1)$ solutions, one near 2 and one near -2.

Step 2: Regular expansion. Assume an expansion

$$x \sim x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots, \qquad \varepsilon \to 0.$$

Then

$$x^2 = x_0^2 + 2\varepsilon x_0 x_1 + \varepsilon^2\left(2x_0x_2 + x_1^2\right) + O(\varepsilon^3),$$

and
$$\varepsilon x = \varepsilon x_0 + \varepsilon^2 x_1 + O(\varepsilon^3), \qquad \frac{\varepsilon^2 x^2}{2} = \frac{\varepsilon^2 x_0^2}{2} + O(\varepsilon^3).$$
Substituting into $x^2 - 4 + \varepsilon x + \frac{\varepsilon^2 x^2}{2} = 0$ and matching powers of ε yields:

Order 1:
$$x_0^2 - 4 = 0 \quad \Rightarrow \quad x_0 = \pm 2.$$
Order ε:
$$2x_0 x_1 + x_0 = 0 \quad \Rightarrow \quad x_1 = -\frac{1}{2}.$$
(Note that x_1 is the same for both branches.)
Order ε^2:
$$2x_0 x_2 + x_1^2 + x_1 + \frac{x_0^2}{2} = 0.$$
Using $x_1 = -\frac{1}{2}$ and $x_0^2 = 4$ gives
$$2x_0 x_2 + \frac{1}{4} - \frac{1}{2} + 2 = 0 \quad \Rightarrow \quad 2x_0 x_2 + \frac{7}{4} = 0 \quad \Rightarrow \quad x_2 = -\frac{7}{8x_0}.$$

Step 3: Two solution branches. For the branch near $x = 2$ (take $x_0 = 2$),
$$x \sim 2 - \frac{\varepsilon}{2} - \frac{7}{16}\varepsilon^2 + O(\varepsilon^3).$$
For the branch near $x = -2$ (take $x_0 = -2$),
$$x \sim -2 - \frac{\varepsilon}{2} + \frac{7}{16}\varepsilon^2 + O(\varepsilon^3).$$

Conclusion. As $\varepsilon \to 0$, the transcendental equation has two real solutions with asymptotic expansions
$$x_+(\varepsilon) \sim 2 - \frac{\varepsilon}{2} - \frac{7}{16}\varepsilon^2 + O(\varepsilon^3), \qquad x_-(\varepsilon) \sim -2 - \frac{\varepsilon}{2} + \frac{7}{16}\varepsilon^2 + O(\varepsilon^3).$$

Example (Beyond-all-orders correction): $x + 1 + \varepsilon\,\mathrm{sech}(x/\varepsilon) = 0$:

Consider the transcendental equation
$$x + 1 + \varepsilon\,\mathrm{sech}\left(\frac{x}{\varepsilon}\right) = 0, \qquad \varepsilon \ll 1,\ \varepsilon > 0.$$
We determine the asymptotic behaviour of its (unique) real solution as $\varepsilon \to 0^+$.

Step 1: Leading-order solution. Since $0 < \mathrm{sech}(z) \le 1$ for all real z, we have
$$0 < \varepsilon\,\mathrm{sech}\left(\frac{x}{\varepsilon}\right) \le \varepsilon \to 0 \qquad (\varepsilon \to 0^+).$$
Hence, to leading order the equation reduces to
$$x_0 + 1 = 0,$$

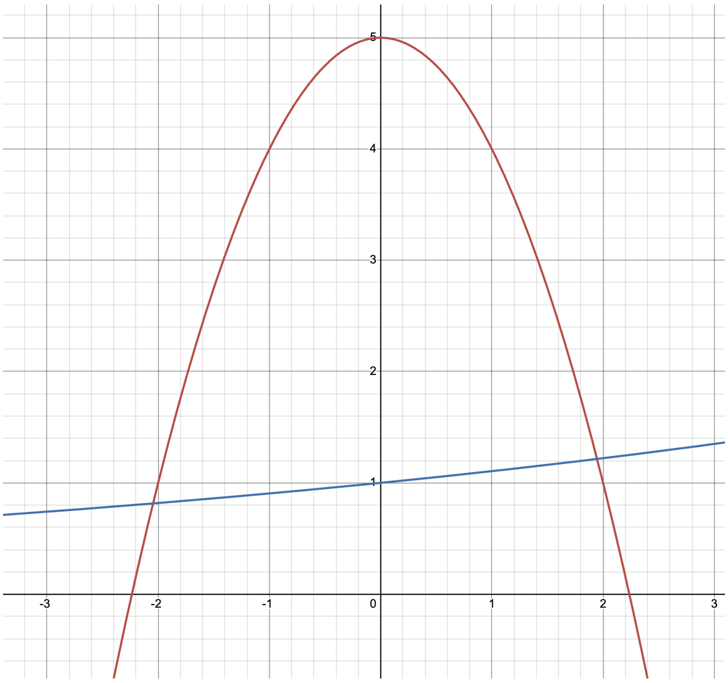


Figure 6: Graph of the functions $f(x, \varepsilon) = 5 - x^2$ and $g(x, \varepsilon) = e^{\varepsilon x}$. For small ε, the roots are close to ± 2.

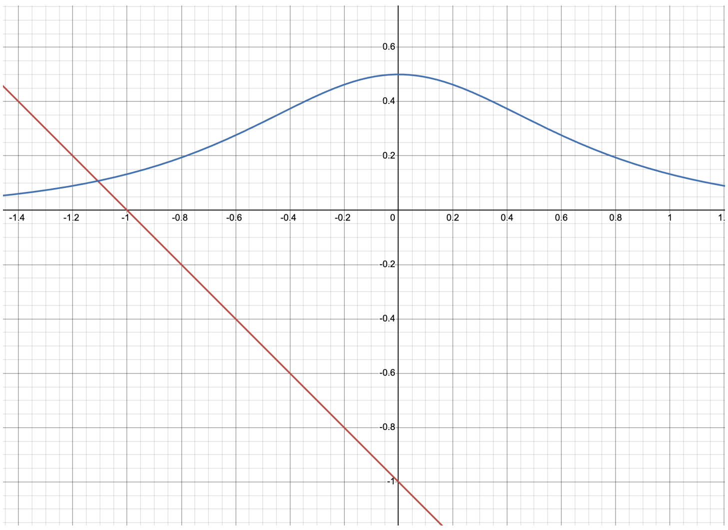


Figure 7: Graph of the functions $f(x, \varepsilon) = -1 - x$ and $g(x, \varepsilon) = \varepsilon \operatorname{sech}(x/\varepsilon)$. For small ε, the root is close to -1.

so
$$x_0 = -1.$$
Thus, the solution satisfies $x \to -1$ as $\varepsilon \to 0^+$.

Step 2: Why a power-law correction cannot balance. Suppose we attempt a correction of the form
$$x \sim -1 + \varepsilon^\alpha x_1, \qquad \alpha > 0.$$
Substituting into the equation gives
$$\varepsilon^\alpha x_1 + \varepsilon \operatorname{sech}\left(\frac{-1+\varepsilon^\alpha x_1}{\varepsilon}\right) = 0.$$
Now observe that
$$\frac{-1+\varepsilon^\alpha x_1}{\varepsilon} = -\varepsilon^{-1} + \varepsilon^{\alpha-1} x_1.$$
As $\varepsilon \to 0^+$, the dominant part is $-\varepsilon^{-1} \to -\infty$, hence
$$\operatorname{sech}\left(\frac{-1+\varepsilon^\alpha x_1}{\varepsilon}\right) \sim \operatorname{sech}(-\varepsilon^{-1}) \sim 2e^{-1/\varepsilon},$$
where we used
$$\operatorname{sech}(z) = \frac{2}{e^z + e^{-z}} \sim 2e^z \quad \text{as } z \to -\infty.$$
Therefore, the second term behaves like $\varepsilon e^{-1/\varepsilon}$. Since
$$\varepsilon e^{-1/\varepsilon} = o(\varepsilon^\alpha) \quad \text{for every } \alpha > 0,$$
no choice of α allows the two terms $\varepsilon^\alpha x_1$ and $\varepsilon e^{-1/\varepsilon}$ to balance. We conclude that the correction to $x = -1$ is not of algebraic order ε^α, but must instead be exponentially small.

Step 3: Introduce an exponentially small correction. Write
$$x \sim -1 + \mu(\varepsilon), \qquad \mu(\varepsilon) \to 0 \text{ as } \varepsilon \to 0^+.$$
Substituting into the equation gives
$$\mu(\varepsilon) + \varepsilon \operatorname{sech}\left(-\varepsilon^{-1} + \frac{\mu(\varepsilon)}{\varepsilon}\right) = 0. \tag{$*$}$$

Step 4: Approximation of the sech term. Since
$$\frac{x}{\varepsilon} = -\varepsilon^{-1} + \frac{\mu(\varepsilon)}{\varepsilon},$$
and $-\varepsilon^{-1} \to -\infty$ as $\varepsilon \to 0^+$, the dominant contribution to the argument of sech comes from $-\varepsilon^{-1}$. Thus,
$$\operatorname{sech}\left(-\varepsilon^{-1} + \frac{\mu(\varepsilon)}{\varepsilon}\right) \sim \operatorname{sech}(-\varepsilon^{-1}) \sim 2e^{-1/\varepsilon},$$

again using $\operatorname{sech}(z) = \frac{2}{e^z+e^{-z}} \sim 2e^z$ as $z \to -\infty$.

Step 5: Solve for $\mu(\varepsilon)$. Substituting this approximation into $(*)$ gives

$$\mu(\varepsilon) + 2\varepsilon e^{-1/\varepsilon} \sim 0,$$

so

$$\boxed{\mu(\varepsilon) \sim -2\varepsilon e^{-1/\varepsilon}.}$$

Conclusion. Hence, the unique real solution satisfies

$$x(\varepsilon) \sim -1 - 2\varepsilon e^{-1/\varepsilon} \qquad (\varepsilon \to 0^+).$$

A more precise estimate is

$$x(\varepsilon) = -1 - 2\varepsilon e^{-1/\varepsilon} + O\big(\varepsilon e^{-2/\varepsilon}\big)\,.$$

This is a typical example where the correction to the leading-order solution is exponentially small and cannot be found by a power-series expansion in ε.

6 Lecture 6: Introduction to the Asymptotic Solution of Differential Equations

The ideas from the previous lectures can be applied to the asymptotic solution of differential equations. We will cover first regular perturbation problems, but will extend this topic in more detail to other types of problems in the next few lectures. We briefly illustrate the main idea with a simple example.

Example (Regular perturbation problem):

Consider the initial value problem

$$\begin{cases} \dfrac{d^2y}{dt^2} = -\dfrac{1}{(1+\varepsilon y)^2}, & t > 0, \\ y(0) = 0, \quad y'(0) = 1. \end{cases}$$

We seek an asymptotic expansion of the solution $y(t)$ as $\varepsilon \to 0$.

Solution: The equation is *weakly nonlinear* because it is nonlinear for $\varepsilon \neq 0$, but reduces to a linear equation when $\varepsilon = 0$. We assume an expansion of the form

$$y(t) \sim y_0(t) + \varepsilon^\alpha y_1(t) + \cdots, \qquad \varepsilon \to 0,$$

where $\alpha > 0$ is initially unknown. We also assume that derivatives may be obtained by differentiating term by term.
For small ε, we use the Taylor expansion

$$\frac{1}{(1+\varepsilon y)^2} = 1 - 2\varepsilon y + O(\varepsilon^2).$$

Substituting this into the differential equation gives

$$y_0''(t) + \varepsilon^\alpha y_1''(t) + \cdots \sim -1 + 2\varepsilon y_0(t) + \cdots$$

The initial conditions yield

$$y_0(0) + \varepsilon^\alpha y_1(0) + \cdots = 0, \qquad y_0'(0) + \varepsilon^\alpha y_1'(0) + \cdots = 1.$$

The terms of the expansion are found by equating like powers of ε.

Order $O(1)$. At leading order,

$$y_0''(t) = -1, \qquad y_0(0) = 0, \quad y_0'(0) = 1.$$

Integrating twice gives

$$y_0(t) = t - \frac{t^2}{2}.$$

Determining α (balancing argument). At the next order, the right-hand side contributes the term $2\varepsilon y_0(t)$, which is $O(\varepsilon)$. Therefore, the left-hand correction term must also be $O(\varepsilon)$, which requires

$$\alpha = 1.$$

Order $O(\varepsilon)$. Equating $O(\varepsilon)$ terms gives

$$y_1''(t) = 2y_0(t) = 2t - t^2, \qquad y_1(0) = 0, \quad y_1'(0) = 0.$$

Integrating twice,

$$y_1'(t) = t^2 - \frac{t^3}{3},$$
$$y_1(t) = \frac{t^3}{3} - \frac{t^4}{12}.$$

Conclusion. Hence, the solution admits the regular asymptotic expansion

$$y(t) \sim t - \frac{t^2}{2} + \varepsilon\left(\frac{t^3}{3} - \frac{t^4}{12}\right) + O(\varepsilon^2), \qquad (\varepsilon \to 0).$$

Conclusion.

The previous example is a rescaled version of the projectile problem from the first class. The previous approximation applies for $0 \leq t \leq t_h$, where t_h is the time at which the solution satisfies $y(t_h) = 0$ (the time when the projectile returns to the ground). More precisely, the expansion

$$y(t) \sim y_0(t) + \varepsilon y_1(t)$$

remains valid uniformly on bounded time intervals up to the hitting time t_h.

Notes:

- The $O(1)$ term $y_0(t)$ is the solution obtained by setting $\varepsilon = 0$ in the differential equation. This is called the *reduced equation*. It corresponds to motion under constant gravity.

- The $O(\varepsilon)$ term $y_1(t)$ contains the first correction due to the nonlinearity. Since

  $$y_1(t) = \frac{t^3}{3} - \frac{t^4}{12} = \frac{t^3}{12}(4 - t),$$

  we see that $y_1(t) > 0$ for $0 < t < 2$. Hence, the correction term increases the height of the projectile for $0 < t < 2$ and, in particular, increases the flight time.

  This agrees with the physics of the problem: the gravitational force decreases with height, so the projectile experiences slightly weaker deceleration and remains aloft longer.

This simple example illustrates that an asymptotic approximation not only provides an approximate solution, but also yields qualitative insight into the behaviour of the system, such as the effect of the nonlinearity on the flight time.

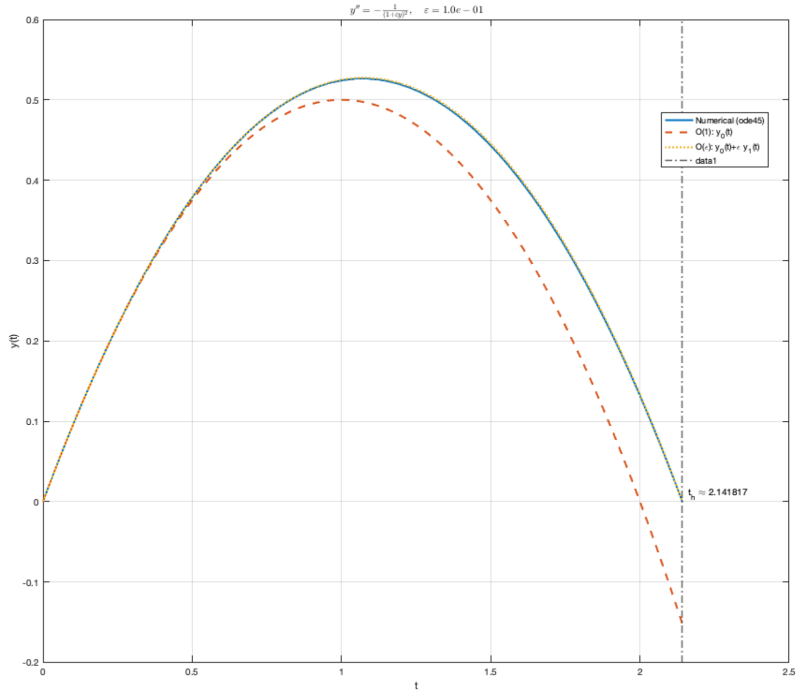


Figure 8: Comparison between the numerical solution of the initial value problem and the asymptotic approximations $y_0(t)$ and $y_0(t) + \varepsilon y_1(t)$ for $\varepsilon = 0.1$.

Example (System of ODEs):

Consider the thermokinetic model

$$\begin{cases} \dfrac{du}{dt} = 1 - u(t)e^{\varepsilon(q(t)-1)}, \\ \dfrac{dq}{dt} = u(t)e^{\varepsilon(q(t)-1)} - q(t), \end{cases} \qquad t > 0,$$

with

$$u(0) = 0, \qquad q(0) = 0.$$

We seek an asymptotic expansion of $(u(t), q(t))$ as $\varepsilon \to 0$.

Solution. For $\varepsilon \to 0$ we expand the exponential:

$$e^{\varepsilon(q-1)} = 1 + \varepsilon(q-1) + O(\varepsilon^2).$$

We assume regular expansions

$$u(t) \sim u_0(t) + \varepsilon u_1(t) + O(\varepsilon^2), \qquad q(t) \sim q_0(t) + \varepsilon q_1(t) + O(\varepsilon^2), \qquad \varepsilon \to 0,$$

and substitute into the system.

Step 1: Substitute and collect powers of ε. First note that

$$u\, e^{\varepsilon(q-1)} = \big(u_0 + \varepsilon u_1 + O(\varepsilon^2)\big)\big(1 + \varepsilon(q_0 - 1) + O(\varepsilon^2)\big) = u_0 + \varepsilon\big(u_1 + u_0(q_0 - 1)\big) + O(\varepsilon^2).$$

Hence, the system becomes

$$u_0' + \varepsilon u_1' + O(\varepsilon^2) = 1 - \Big[u_0 + \varepsilon\big(u_1 + u_0(q_0 - 1)\big)\Big] + O(\varepsilon^2),$$

$$q_0' + \varepsilon q_1' + O(\varepsilon^2) = \Big[u_0 + \varepsilon\big(u_1 + u_0(q_0 - 1)\big)\Big] - \big(q_0 + \varepsilon q_1\big) + O(\varepsilon^2).$$

The initial conditions give

$$u_0(0) = 0, \quad q_0(0) = 0, \qquad u_1(0) = 0, \quad q_1(0) = 0.$$

Order $O(1)$. Equating $O(1)$ terms yields the linear system

$$u_0' = 1 - u_0, \qquad q_0' = u_0 - q_0, \qquad u_0(0) = 0, \quad q_0(0) = 0.$$

Solve the first equation:

$$u_0' + u_0 = 1 \quad \Rightarrow \quad u_0(t) = 1 - e^{-t}.$$

Substitute u_0 into the second equation:

$$q_0' + q_0 = 1 - e^{-t}.$$

Using an integrating factor e^t,

$$(e^t q_0)' = e^t(1 - e^{-t}) = e^t - 1,$$

so

$$e^t q_0(t) = \int_0^t (e^s - 1)\, ds = (e^t - 1) - t,$$

and therefore

$$q_0(t) = 1 - e^{-t} - te^{-t}.$$

Order $O(\varepsilon)$. Equating $O(\varepsilon)$ terms yields

$$u_1' = -\big(u_1 + u_0(q_0 - 1)\big), \qquad q_1' = \big(u_1 + u_0(q_0 - 1)\big) - q_1,$$

with $u_1(0) = 0$ and $q_1(0) = 0$. It is convenient to rewrite the forcing term explicitly. Since

$$q_0(t) - 1 = -e^{-t}(1 + t),$$

we have

$$u_0(q_0 - 1) = (1 - e^{-t})\big(-e^{-t}(1 + t)\big) = -(1 + t)e^{-t} + (1 + t)e^{-2t}.$$

Solve for u_1. The equation for u_1 is

$$u_1' + u_1 = -u_0(q_0 - 1) = (1 + t)e^{-t} - (1 + t)e^{-2t}.$$

Multiply by e^t:

$$(e^t u_1)' = (1 + t) - (1 + t)e^{-t}.$$

Integrate from 0 to t and use $u_1(0) = 0$:

$$e^t u_1(t) = \int_0^t (1+s)\,ds - \int_0^t (1+s)e^{-s}\,ds.$$

Compute the integrals:

$$\int_0^t (1+s)\,ds = t + \frac{t^2}{2}, \qquad \int_0^t (1+s)e^{-s}\,ds = 2 - (t+2)e^{-t}.$$

Hence,

$$e^t u_1(t) = t + \frac{t^2}{2} - 2 + (t+2)e^{-t},$$

and therefore

$$u_1(t) = e^{-t}\left(t + \frac{t^2}{2} - 2\right) + (t+2)e^{-2t}.$$

Solve for q_1. From the q_1 equation,

$$q_1' + q_1 = u_1 + u_0(q_0 - 1).$$

Using the expressions above,

$$u_1 + u_0(q_0 - 1) = \left[e^{-t}\left(t + \frac{t^2}{2} - 2\right) + (t+2)e^{-2t}\right] + \left[-(1+t)e^{-t} + (1+t)e^{-2t}\right].$$

Simplifying,

$$u_1 + u_0(q_0 - 1) = e^{-t}\left(\frac{t^2}{2} - 3\right) + (2t+3)e^{-2t}.$$

Multiply by e^t:

$$(e^t q_1)' = \left(\frac{t^2}{2} - 3\right) + (2t+3)e^{-t}.$$

Integrate from 0 to t and use $q_1(0) = 0$:

$$e^t q_1(t) = \int_0^t \left(\frac{s^2}{2} - 3\right)\,ds + \int_0^t (2s+3)e^{-s}\,ds.$$

Compute:

$$\int_0^t \left(\frac{s^2}{2} - 3\right)\,ds = \frac{t^3}{6} - 3t, \qquad \int_0^t (2s+3)e^{-s}\,ds = 5 - (2t+5)e^{-t}.$$

Thus,

$$e^t q_1(t) = \frac{t^3}{6} - 3t + 5 - (2t+5)e^{-t},$$

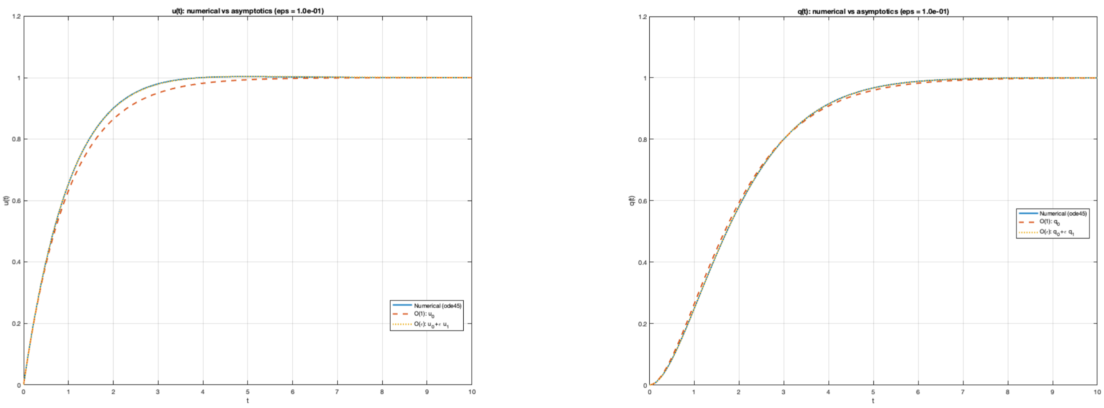
and hence

$$q_1(t) = e^{-t}\left(\frac{t^3}{6} - 3t + 5\right) - (2t+5)e^{-2t}.$$

Conclusion. Combining the terms, the regular asymptotic expansions as $\varepsilon \to 0$ are

$$u(t) \sim u_0(t) + \varepsilon u_1(t) = \left(1 - e^{-t}\right) + \varepsilon\left[e^{-t}\left(t + \frac{t^2}{2} - 2\right) + (t+2)e^{-2t}\right] + O(\varepsilon^2),$$

$$q(t) \sim q_0(t) + \varepsilon q_1(t) = \left(1 - e^{-t} - te^{-t}\right) + \varepsilon\left[e^{-t}\left(\frac{t^3}{6} - 3t + 5\right) - (2t+5)e^{-2t}\right] + O(\varepsilon^2)$$



(a) Comparison between the numerical solution for $u(t)$ and the asymptotic approximations $u_0(t)$ and $u_0(t) + \varepsilon u_1(t)$ for $\varepsilon = 0.1$.

(b) Comparison between the numerical solution for $q(t)$ and the asymptotic approximations $q_0(t)$ and $q_0(t) + \varepsilon q_1(t)$ for $\varepsilon = 0.1$.

Figure 9: Comparison between the numerical solution of the system of ODEs and the asymptotic approximations for $\varepsilon = 0.1$.

7 Lecture 7: Matched Asymptotic Expansions

In the previous lectures we studied *regular perturbation problems*, where solutions admit power–series expansions in ε. However, many problems of interest are *singular perturbation problems*, where the solution does not admit a regular expansion and instead exhibits multiple scales or *boundary layers*.

In such situations, a regular expansion is no longer sufficient. We will use now the method of *matched asymptotic expansions* to construct an approximation that is valid throughout a whole interval.

The best way to understand the method is through an example.

A motivating example

Consider the boundary value problem

$$\begin{cases} \varepsilon y''(x) + 2y'(x) + 2y(x) = 0, & x \in (0,1), \\ y(0) = 0, \quad y(1) = 1, \end{cases}$$

where $\varepsilon > 0$ is small.

What happens if $\varepsilon = 0$?

If we formally set $\varepsilon = 0$, the equation becomes

$$2y_0'(x) + 2y_0(x) = 0.$$

Notice that the equation has dropped from second order to first order. This loss of order is the signature of a *singular perturbation problem*.

Solving,

$$y_0(x) = Ce^{-x}.$$

There is only one constant C, but we have two boundary conditions. So the reduced problem cannot satisfy both boundary conditions simultaneously.

Let us examine what happens.

If we impose $y(0) = 0$, then $C = 0$, which gives the trivial solution $y_0 \equiv 0$. If instead we impose $y(1) = 1$, then $C = e$, giving

$$y_0(x) = e^{1-x}.$$

Neither choice satisfies both boundary conditions.

This is our first indication that something more subtle is happening.

Note:

We could have also started by looking at a regular expansion. That is, we assume that the solution can be expanded in a regular power series in ε:

$$y(x) \sim y_0(x) + \varepsilon y_1(x) + O(\varepsilon^2), \quad \varepsilon \to 0^+.$$

Substituting this expansion into the ODE we obtain

$$\varepsilon y_0''(x) + 2y_0'(x) + 2y_0(x) + \varepsilon\left(y_1''(x) + 2y_1'(x) + 2y_1(x)\right) + O(\varepsilon^2) = 0.$$

Equating the $O(1)$ terms gives

$$2y_0'(x) + 2y_0(x) = 0.$$

This is a first-order linear ODE, and its general solution is

$$y_0(x) = Ce^{-x},$$

where C is a constant to be determined. Observe that there is only one arbitrary constant but two boundary conditions. Which boundary condition should we apply to determine C? If we apply $y(0) = 0$, then we get $C = 0$, which gives the trivial solution $y_0(x) = 0$. This does not satisfy the second boundary condition $y(1) = 1$. On the other hand, if we apply $y(1) = 1$, then we get $C = e$, which gives $y_0(x) = e^{1-x}$. This does not satisfy the first boundary condition $y(0) = 0$. This indicates that the regular expansion is not valid across the entire interval $(0, 1)$.

Understanding the behaviour (using plots)

This particular example is simple enough to be solved explicitly, which allows us to understand what is happening in this problem.

Exercise:

Find the exact solution of the boundary value problem.

Solution outline.

The characteristic equation is

$$\varepsilon r^2 + 2r + 2 = 0,$$

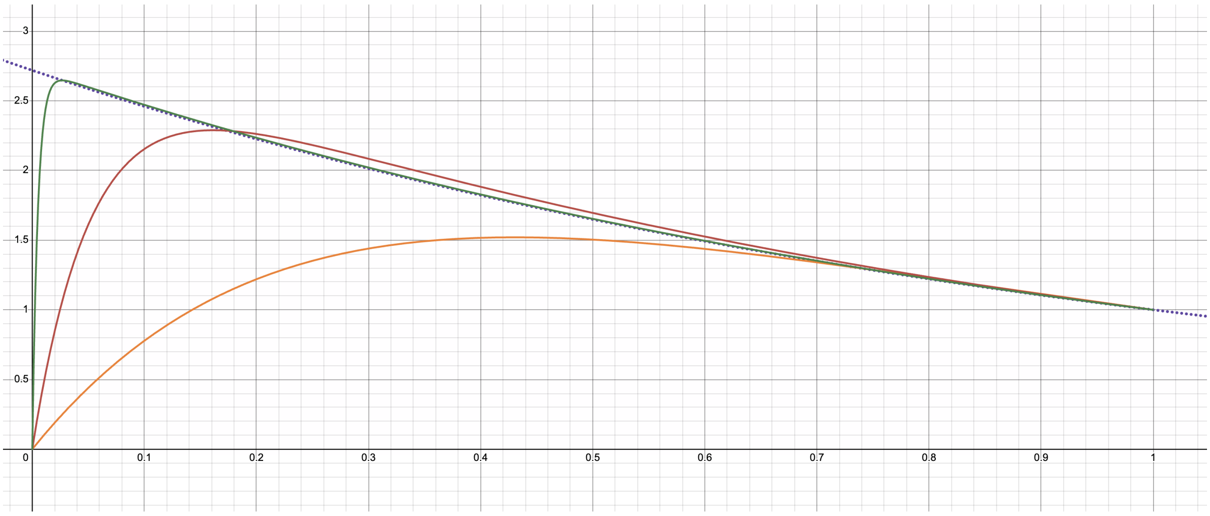


Figure 10: Plot of the exact solution for $\varepsilon = 0.4$, $\varepsilon = 0.1$, and $\varepsilon = 0.01$. For small ε, the solution is close to e^{1-x} for most of the interval, but changes very rapidly near $x = 0$. Also plot is the function e^{1-x} in the background for comparison.

with roots

$$r_{1,2} = \frac{-1 \pm \sqrt{1-2\varepsilon}}{\varepsilon}.$$

For small ε, these roots are real and distinct. Hence,

$$y(x) = Ae^{r_1 x} + Be^{r_2 x}.$$

Applying the boundary conditions,

$$A + B = 0, \qquad Ae^{r_1} + Be^{r_2} = 1.$$

Solving,

$$A = \frac{1}{e^{r_1} - e^{r_2}}, \qquad B = -\frac{1}{e^{r_1} - e^{r_2}}.$$

Thus,

$$y(x) = \frac{e^{r_1 x} - e^{r_2 x}}{e^{r_1} - e^{r_2}}.$$

We plot and show this solution for decreasing values of ε in Figure 10.

Observation from the plots.
For small ε, the solution:

- is close to e^{1-x} for most of the interval,
- but changes very rapidly near $x = 0$.

Important:

Key observation. The boundary layer has width of about ε.

This rapid change occurs in a very small region near $x = 0$. That region is called a *boundary layer*. This difference in behaviour of the function along the domain motivates the following strategy:

- Construct an **outer solution** (outside the boundary layer) valid away from $x = 0$.
- Construct an **inner solution** (inside the boundary layer) valid near $x = 0$.
- Match them to obtain a uniformly valid approximation.
- Construct a **composite solution** that is valid throughout the whole interval.

Step 1: Outer solution

Away from $x = 0$, we assume that the solution can be expanded in powers of ε. In other words,

$$y(x) \sim y_0(x) + \varepsilon y_1(x) + \cdots .$$

Substituting this into the ODE we obtain

$$\varepsilon \left(y_0''(x) + \varepsilon y_1''(x) + \cdots \right) + 2 \left(y_0'(x) + \varepsilon y_1'(x) + \cdots \right) + 2 \left(y_0(x) + \varepsilon y_1(x) + \cdots \right) = 0.$$

The $O(1)$ equation is therefore

$$y_0'(x) + y_0(x) = 0,$$

and the general solution is

$$y_0(x) = Ce^{-x}.$$

Since the boundary layer is near $x = 0$, the outer solution should satisfy the boundary condition at the *other* end:

$$y(1) = 1.$$

This gives

$$C = e, \qquad y_{\text{outer}}(x) = e^{1-x}.$$

This function approximates the solution well except near $x = 0$.

Step 2: Inner (boundary layer) solution

We now zoom into the thin region near $x = 0$. The width of the boundary layer shrinks as $\varepsilon \to 0$. To study what happens in this region, we would like to find a different scale for x that captures the rapid change. To do this, we need to formulate the problem in terms of a new variable that is appropriate for the boundary layer.

First, we introduce a 'stretched' variable

$$\xi = \frac{x}{\delta},$$

where δ is the unknown thickness of the boundary layer.

Using

$$\frac{d}{dx} = \frac{d\xi}{dx}\frac{d}{d\xi} = \frac{1}{\delta}\frac{d}{d\xi}, \qquad \frac{d^2}{dx^2} = \frac{d}{dx}\frac{d}{dx} = \frac{1}{\delta^2}\frac{d^2}{d\xi^2},$$

the ODE becomes

$$\varepsilon\frac{1}{\delta^2}y'' + 2\frac{1}{\delta}y' + 2y = 0.$$

To retain the highest derivative in the leading-order equation, we need to balance the equation. We could try to balance out the different terms. If we try to balance the first and third terms, for example, we will get that the second term is $O(\varepsilon^{-1/2})$, but this is not getting smaller as $\varepsilon \to 0$. The only way to balance the equation is to balance the first and second terms:

$$\frac{\varepsilon}{\delta^2} \sim \frac{1}{\delta} \quad \Rightarrow \quad \delta \sim \varepsilon.$$

Thus, we set

$$\xi = \frac{x}{\varepsilon}.$$

The ODE becomes

$$y''(\xi) + 2y'(\xi) + 2\varepsilon y(\xi) = 0, \qquad 0 < \xi < \infty.$$

As $\varepsilon \to 0$, the leading-order inner equation is

$$y''(\xi) + 2y'(\xi) = 0.$$

Solving,

$$y_{\text{inner}}(\xi) = C_1 + C_2 e^{-2\xi}.$$

Applying the boundary condition at $x = 0$ (i.e. $\xi = 0$),

$$C_1 + C_2 = 0,$$

so

$$y_{\text{inner}}(\xi) = C_2(e^{-2\xi} - 1).$$

Step 3: Matching

Our approximation consists of two different expansions, and each applies to a different part of the interval. This situation is sketched in Figure 11. When coming out of the boundary layer, the inner solution approaches a constant value C_2. Similarly, the outer solution approaches a constant value e as it enters the boundary layer. There is a transition region, that is called the *overlap region*, where the two approximations are both constant. Given that they are approximations of the same function, then we need to require that the inner and outer expansions are equal in this region.

To match the two solutions, we examine the overlap region:

$$x \to 0 \quad \text{and} \quad \xi \to \infty.$$

We require that

$$\lim_{x\to 0} y_{\text{outer}}(x) = \lim_{\xi\to\infty} y_{\text{inner}}(\xi).$$

Outer limit:

$$y_{\text{outer}}(x) \to e.$$

Inner limit:

$$y_{\text{inner}}(\xi) \to -C_2.$$

Matching gives

$$-e = C_2.$$

Hence,

$$y_{\text{inner}}(\xi) = e(1 - e^{-2\xi}).$$

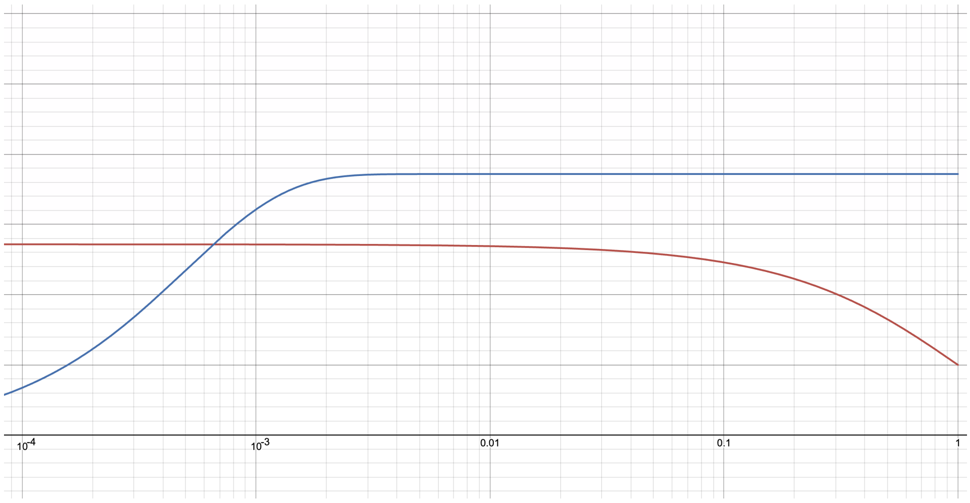
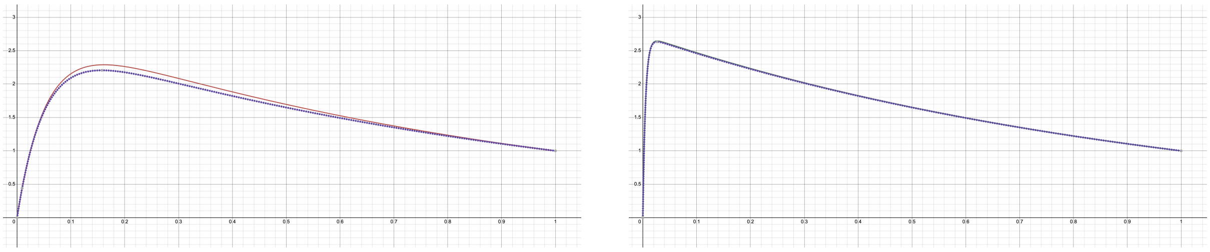


Figure 11: Sketch of the matching procedure. The outer solution is valid away from $x = 0$, while the inner solution is valid near $x = 0$. The two solutions are matched in the overlap region where $x \to 0$ and $\xi \to \infty$.



(a) Comparison between the exact solution and the composite solution for $\varepsilon = 0.1$.

(b) Comparison between the exact solution and the composite solution for $\varepsilon = 0.01$.

Figure 12: Comparison between the exact solution of the boundary value problem and the composite solution for decreasing values of ε.

Step 4: Composite solution

Our solution consists of two pieces, one that applies near the boundary layer and one that applies away from it. Because neither can be used over the entire interval, they are not uniformly valid for $0 \leq x \leq 1$. As a consequence of the matching condition, we need to subtract the part that is common to both. The result is the *composite solution*:

$$y_{\text{comp}}(x) = y_{\text{outer}}(x) + y_{\text{inner}}(x/\varepsilon) - L.$$

Where L can be taken either as $y_{\text{outer}}(0)$ or $y_{\text{inner}}(\infty)$. Both are equal to e by the matching condition.

This gives

$$y_{\text{comp}}(x) = e^{1-x} + e(1 - e^{-2x/\varepsilon}) - e = e^{1-x} - e^{1-2x/\varepsilon}.$$

We now plot the exact solution and the composite solution for decreasing values of ε in Figure 12.

We observe that:

- the composite solution matches the exact solution throughout $(0, 1)$ as $\varepsilon \to 0$,

- the boundary layer thickness is $O(\varepsilon)$.
- Note that the composite solution satisfies the boundary condition at $x = 0$ exactly, but only satisfies the boundary condition at $x = 1$ asymptotically. This is a common feature of composite solutions.

8 Lecture 8: More on Matched Asymptotic Expansions

In the previous lecture we constructed a leading-order matched asymptotic approximation for

$$\begin{cases} \varepsilon y'' + 2y' + 2y = 0, & 0 < x < 1, \\ y(0) = 0, \quad y(1) = 1, \end{cases}$$

with $0 < \varepsilon \ll 1$.

We now refine that approximation by constructing the $O(\varepsilon)$ terms. The second term is important as it gives a measure of the error. We usually call it the *first correction*.

Preliminary remarks on matching

Before proceeding, let us clarify two important points.

- The boundary layer thickness shrinks to zero as $\varepsilon \to 0$, but the total change in the solution across the layer remains $O(1)$. This behaviour is typical in singular perturbation problems.
- Matching does *not* mean that the inner and outer approximations must intersect. For example, the functions

  $$1 - e^{-x/\varepsilon} \quad \text{and} \quad 1 + x$$

  are inner and outer approximations of

  $$y = 1 + x - e^{-x/\varepsilon},$$

  yet they never intersect. Matching concerns agreement of asymptotic behaviour in an overlap region, not intersection.

The simple matching procedure used previously (equating limits) works when the limits exist. However, when constructing higher-order terms, a more systematic matching procedure is required.

1. Outer expansion to $O(\varepsilon)$

We assume an outer expansion of the form

$$y(x) \sim y_0(x) + \varepsilon y_1(x) + \cdots .$$

Substituting into the ODE:

$$\varepsilon(y_0'' + \varepsilon y_1'' + \cdots) + 2(y_0' + \varepsilon y_1' + \cdots) + 2(y_0 + \varepsilon y_1 + \cdots) = 0.$$

Order $O(1)$.
From before we know that

$$y_0(x) = e^{1-x}.$$

Order $O(\varepsilon)$.
Collecting $O(\varepsilon)$ terms:

$$y_0'' + 2y_1' + 2y_1 = 0.$$

Since $y_0'' = e^{1-x}$, we obtain

$$2y_1' + 2y_1 = -e^{1-x}.$$

Divide by 2:

$$y_1' + y_1 = -\tfrac{1}{2}e^{1-x}.$$

This is a first-order linear equation.
Using the integrating factor e^x:

$$\frac{d}{dx}(e^x y_1) = -\tfrac{1}{2}e.$$

Integrating:

$$e^x y_1 = -\tfrac{1}{2}ex + C_1.$$

Thus

$$y_1(x) = e^{-x}\big(C_1 - \tfrac{1}{2}ex\big).$$

To determine C_1, we impose the boundary condition at $x = 1$:

$$y_1(1) = 0.$$

This gives

$$0 = e^{-1}\big(C_1 - \tfrac{1}{2}e\big) \quad \Rightarrow \quad C_1 = \tfrac{1}{2}e.$$

Therefore,

$$\boxed{y_1(x) = \frac{e}{2}e^{-x}(1 - x).}$$

So the outer expansion to first order is

$$y_{\text{outer}} \sim e^{1-x} + \varepsilon\frac{e}{2}e^{-x}(1 - x).$$

2. Inner expansion to $O(\varepsilon)$

Recall the stretched variable

$$\xi = \frac{x}{\varepsilon}.$$

The equation becomes

$$y'' + 2y' + 2\varepsilon y = 0.$$

Assume

$$y(\xi) \sim Y_0(\xi) + \varepsilon Y_1(\xi) + \cdots .$$

Order $O(\varepsilon)$.

$$Y_1'' + 2Y_1' = -2Y_0.$$

Using the leading-order inner solution from the previous lecture,

$$Y_0(\xi) = e(1 - e^{-2\xi}),$$

we obtain

$$Y_1'' + 2Y_1' = -2e(1 - e^{-2\xi}).$$

Let $Z = Y_1'$. Then

$$Z' + 2Z = -2e(1 - e^{-2\xi}).$$

With integrating factor $e^{2\xi}$,

$$\frac{d}{d\xi}\big(e^{2\xi}Z\big) = -2e(e^{2\xi} - 1).$$

Integrating,

$$e^{2\xi}Z = -e\,e^{2\xi} + 2e\xi + C,$$

so

$$Y_1' = -e + 2e\xi e^{-2\xi} + Ce^{-2\xi}.$$

Integrating again, we use

$$\int 2e\xi e^{-2\xi}\,d\xi = -\,e\xi e^{-2\xi} - \frac{e}{2}e^{-2\xi}, \qquad \int Ce^{-2\xi}\,d\xi = -\frac{C}{2}e^{-2\xi},$$

to obtain

$$Y_1(\xi) = -e\xi - e\xi e^{-2\xi} - \frac{e}{2}e^{-2\xi} - \frac{C}{2}e^{-2\xi} + K,$$

where K is a constant.

Now impose the boundary condition at $x = 0$. Since

$$y(0) = Y_0(0) + \varepsilon Y_1(0) + \cdots = 0 \quad \text{and} \quad Y_0(0) = 0,$$

we must have

$$Y_1(0) = 0.$$

Substituting $\xi = 0$ gives

$$0 = 0 - 0 - \frac{e}{2} - \frac{C}{2} + K \quad \Rightarrow \quad K = \frac{e+C}{2}.$$

Therefore,

$$\boxed{Y_1(\xi) = \frac{e+C}{2}\left(1 - e^{-2\xi}\right) - e\xi\left(1 + e^{-2\xi}\right).}$$

3. Matching at $O(\varepsilon)$ using a general overlap scaling

At leading order we matched by simply taking limits:

$$\lim_{x \to 0} y_0(x) = \lim_{\xi \to \infty} Y_0(\xi),$$

because both limits existed and were finite constants.

At $O(\varepsilon)$ this approach fails. Indeed, from the previous section

$$Y_1(\xi) \sim \frac{e}{2} - e\xi \quad (\xi \to \infty),$$

which diverges as $\xi \to \infty$. Therefore,

$$\lim_{\xi \to \infty} Y_1(\xi) \quad \text{does not exist.}$$

Hence, we cannot match by equating limits of y_1 and Y_1.

Instead, we must match *asymptotic expansions* in an *overlap region* where both the inner and outer approximations are simultaneously valid.

Step 1: Introduce a general overlap scaling.
The outer solution is valid for $x = O(1)$, the inner solution is valid for $x = O(\varepsilon)$. Therefore, the overlap region must satisfy

$$\varepsilon \ll x \ll 1.$$

We introduce a general intermediate scaling

$$x = \varepsilon^{\beta} x_{\eta}, \qquad 0 < \beta < 1.$$

This guarantees:

$$x \to 0 \quad \text{(so the outer expansion can be expanded about } x = 0\text{)},$$

and

$$\xi = \frac{x}{\varepsilon} = \varepsilon^{\beta - 1} x_{\eta} \to \infty \quad \text{(so the inner expansion can be expanded for large } \xi\text{)}.$$

We now determine β by matching.

Step 2: Outer expansion in the overlap region.

Recall

$$y_{\text{outer}}(x) = e^{1-x} + \varepsilon \frac{e}{2} e^{-x}(1-x).$$

Expand $y_0(x) = e^{1-x}$ *near* $x = 0$*:*

$$e^{1-x} = e\left(1 - x + \frac{x^2}{2} + O(x^3)\right).$$

Substitute $x = \varepsilon^{\beta} x_{\eta}$:

$$y_0 = e - e\,\varepsilon^{\beta} x_{\eta} + \frac{e}{2}\varepsilon^{2\beta} x_{\eta}^2 + O(\varepsilon^{3\beta}).$$

Expand $\varepsilon y_1(x)$*.*
From earlier,

$$y_1(x) = \frac{e}{2} - ex + O(x^2).$$

Thus,

$$\varepsilon y_1(x) = \frac{e}{2}\varepsilon - e\varepsilon^{1+\beta} x_{\eta} + O(\varepsilon^{1+2\beta}).$$

Combine:

$$y_{\text{outer}} = e - e\,\varepsilon^{\beta} x_{\eta} + \frac{e}{2}\varepsilon + \frac{e}{2}\varepsilon^{2\beta} x_{\eta}^2 + \text{higher order terms.}$$

Step 3: Inner expansion in the overlap region.
The inner expansion is

$$y_{\text{inner}} = Y_0(\xi) + \varepsilon Y_1(\xi), \qquad \xi = \frac{x}{\varepsilon}.$$

Under the overlap scaling,

$$\xi = \varepsilon^{\beta-1} x_{\eta} \to \infty.$$

For large ξ we previously found:

$$Y_0(\xi) \to e, \qquad Y_1(\xi) \sim \frac{e}{2} - e\xi.$$

Therefore,

$$y_{\text{inner}} \sim e + \varepsilon\left(\frac{e}{2} - e\xi\right).$$

Substitute $\xi = \varepsilon^{\beta-1} x_{\eta}$:

$$y_{\text{inner}} = e + \frac{e}{2}\varepsilon - e\varepsilon\varepsilon^{\beta-1} x_{\eta}.$$

Simplifying,

$$\boxed{y_{\text{inner}} = e - e\,\varepsilon^{\beta} x_{\eta} + \frac{e}{2}\varepsilon.}$$

Step 4: Determine β by matching.
Comparing inner and outer overlap expansions:
Outer:

$$e - e\,\varepsilon^{\beta} x_{\eta} + \frac{e}{2}\varepsilon + \frac{e}{2}\varepsilon^{2\beta} x_{\eta}^2 + \cdots$$

Inner:

$$e - e\,\varepsilon^{\beta} x_\eta + \frac{e}{2}\varepsilon.$$

These agree in the first three terms.

The extra outer term

$$\frac{e}{2}\varepsilon^{2\beta} x_\eta^2$$

must be smaller than $O(\varepsilon)$ in order not to spoil matching.

Thus, we require

$$\varepsilon^{2\beta} \ll \varepsilon \quad \Longrightarrow \quad 2\beta > 1.$$

The *critical* value occurs when

$$2\beta = 1, \qquad \beta = \frac{1}{2}.$$

Hence, the natural overlap scaling is

$$\boxed{x = \sqrt{\varepsilon}\, x_\eta.}$$

This is not chosen arbitrarily — it emerges from balancing powers of ε during matching.

Step 5: The common part.
Using $\beta = \frac{1}{2}$, the common part of both expansions is

$$\boxed{y_{\text{common}} = e + \frac{e}{2}\varepsilon - e\sqrt{\varepsilon}\, x_\eta.}$$

Remark.
The appearance of $\sqrt{\varepsilon}$ does not mean that the boundary layer thickness is $O(\sqrt{\varepsilon})$. The boundary layer remains $O(\varepsilon)$.

The $\sqrt{\varepsilon}$ arises purely from the *overlap scaling*, which describes the intermediate region between the inner and outer solutions.

This completes the matching at $O(\varepsilon)$ using a systematic overlap analysis.

3b. Matching at $O(\varepsilon)$ (revisited)

At leading order, matching consisted of equating limits:

$$\lim_{x\to 0} y_0(x) = \lim_{\xi\to\infty} Y_0(\xi).$$

This worked because both limits existed and were finite constants.

At $O(\varepsilon)$ this strategy no longer works. Indeed, from the previous calculation,

$$Y_1(\xi) \sim -e\xi + \text{constant} \qquad (\xi \to \infty),$$

so the limit of $Y_1(\xi)$ as $\xi \to \infty$ does not exist. Therefore, we cannot match by simply equating limits of y_1 and Y_1.

Instead, we must match *asymptotic expansions in the overlap region.*

The overlap region.

The overlap region is characterised by

$$x \to 0, \qquad \xi = \frac{x}{\varepsilon} \to \infty.$$

Equivalently, we may write

$$x = \varepsilon\xi, \quad \text{with } \xi \to \infty.$$

In this region, both the outer and inner expansions are valid. To perform matching we:

- expand the outer solution for small x,
- rewrite it in terms of ξ using $x = \varepsilon\xi$,
- expand the inner solution for large ξ,
- compare like powers of ε.

Outer expansion in the overlap region.
Recall

$$y_{\text{outer}}(x) = y_0(x) + \varepsilon y_1(x).$$

We first expand each term for small x.

Expansion of $y_0(x)$.

$$y_0(x) = e^{1-x} = e\, e^{-x}.$$

Using the Taylor expansion of e^{-x} near $x = 0$,

$$e^{-x} = 1 - x + \frac{x^2}{2} + O(x^3),$$

we obtain

$$y_0(x) = e\left(1 - x + \frac{x^2}{2} + O(x^3)\right).$$

Now substitute $x = \varepsilon\xi$:

$$y_0(\varepsilon\xi) = e - e\varepsilon\xi + O(\varepsilon^2).$$

(The term $\frac{e}{2}\varepsilon^2\xi^2$ is $O(\varepsilon^2)$ and therefore beyond the order we are matching.)

Expansion of $y_1(x)$.
Recall

$$y_1(x) = \frac{e}{2}e^{-x}(1 - x).$$

Expanding for small x:

$$e^{-x} = 1 - x + \frac{x^2}{2} + O(x^3),$$

so

$$e^{-x}(1 - x) = 1 - 2x + O(x^2).$$

Hence,

$$y_1(x) = \frac{e}{2} - ex + O(x^2).$$

Now substitute $x = \varepsilon\xi$:

$$y_1(\varepsilon\xi) = \frac{e}{2} - e\varepsilon\xi + O(\varepsilon^2).$$

Multiplying by ε,

$$\varepsilon y_1(\varepsilon\xi) = \varepsilon\frac{e}{2} + O(\varepsilon^2).$$

(Notice that the term $-e\varepsilon^2\xi$ is $O(\varepsilon^2)$ and does not contribute at $O(\varepsilon)$.)

Combining the two contributions.
Adding $y_0(\varepsilon\xi)$ and $\varepsilon y_1(\varepsilon\xi)$,

$$y_{\text{outer}} \sim e + \varepsilon\Big(-e\xi + \frac{e}{2}\Big) + O(\varepsilon^2).$$

Inner expansion in the overlap region.
We now expand the inner solution for large ξ.
From the previous lecture we already have $A = e$. Thus,

$$Y_0(\xi) = e(1 - e^{-2\xi}) \quad \Rightarrow \quad Y_0(\xi) \to e \quad (\xi \to \infty).$$

From the $O(\varepsilon)$ calculation,

$$Y_1(\xi) = \frac{e + C}{2}(1 - e^{-2\xi}) - e\xi(1 + e^{-2\xi}).$$

As $\xi \to \infty$, $e^{-2\xi} \to 0$, so

$$Y_1(\xi) \sim -e\xi + \frac{e + C}{2}.$$

Therefore, in the overlap region,

$$y_{\text{inner}} \sim e + \varepsilon\left(-e\xi + \frac{e + C}{2}\right).$$

Matching.
We now compare the overlap expansions:

$$\text{Outer:} \quad e + \varepsilon\Big(-e\xi + \frac{e}{2}\Big),$$

$$\text{Inner:} \quad e + \varepsilon\left(-e\xi + \frac{e + C}{2}\right).$$

The $O(1)$ terms already agree. The coefficients of $-\varepsilon e\xi$ also agree.
Matching the constant $O(\varepsilon)$ terms gives

$$\frac{e}{2} = \frac{e + C}{2},$$

and hence

$$C = 0.$$

This determines the remaining constant in the inner expansion.

4. Composite approximation and the common part

With $C = 0$, the approximations are

$$y_{\text{out}}(x) = e^{1-x} + \varepsilon \frac{e}{2} e^{-x}(1-x),$$

and, with $\xi = x/\varepsilon$,

$$y_{\text{in}}(x) = e(1 - e^{-2x/\varepsilon}) + \varepsilon \frac{e}{2}(1 - e^{-2x/\varepsilon}) - ex(1 + e^{-2x/\varepsilon}).$$

In the overlap region ($x = \varepsilon\xi$, $\xi \to \infty$), both reduce to the same common part

$$y_{\text{common}} = e + \varepsilon \left(-e\xi + \frac{e}{2} \right) = e - ex + \varepsilon \frac{e}{2}.$$

Hence, the composite approximation is

$$y_{\text{comp}} = y_{\text{out}} + y_{\text{in}} - y_{\text{common}},$$

that is,

$$y_{\text{comp}}(x) = e^{1-x} + \varepsilon \frac{e}{2} e^{-x}(1-x) - e\, e^{-2x/\varepsilon} - ex\, e^{-2x/\varepsilon} - \varepsilon \frac{e}{2} e^{-2x/\varepsilon}.$$

9 Lecture 9: Interior and Boundary Layers

For example, there are problems where the jump occurs in the interval $0 < x < 1$ instead of at the boundary, and in that case the layer is called an *interior layer*. They also arise in IVPs, PDEs, etc. Last lecture we only computed the first term.

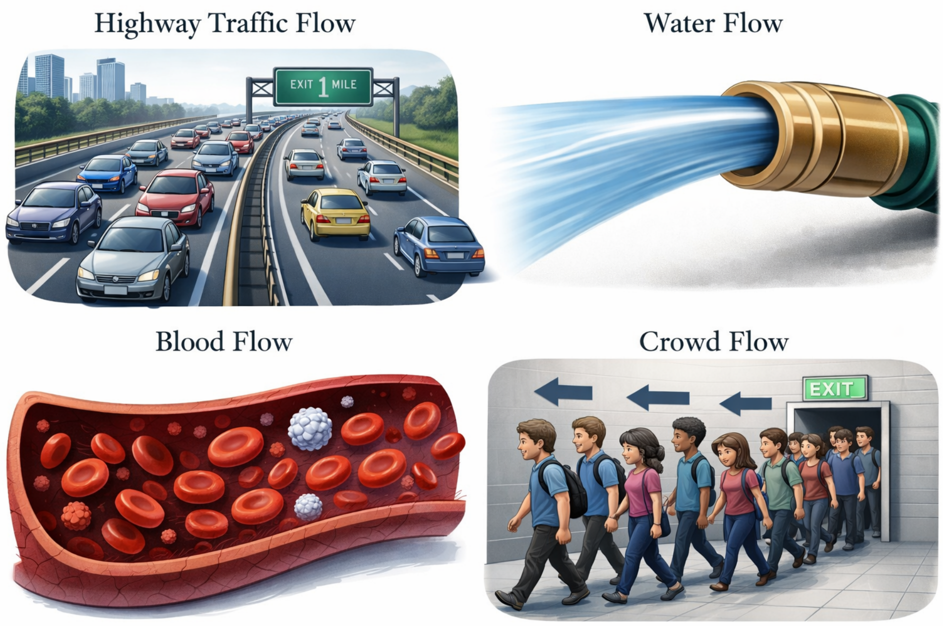


Figure 13: Sketch of traffic flow on a highway. The density ρ and flux J are defined at each point along the road and at each time.

10 Lecture 13: Traffic Flow

10.1 Introduction

Traffic flow provides a classical example of a **partial differential equation (PDE)** arising from a real-world problem.

We consider the motion of objects along a (effectively) one-dimensional pathway, such as cars moving along a highway. Rather than tracking each individual vehicle, we adopt a **continuum description**, where quantities are averaged and treated as continuous functions of space and time.

Remark. Although the ideas apply to many contexts (e.g. fluids, crowds, or data flow), we focus on cars on a highway because this setting is intuitive and easy to interpret.

10.2 Continuum Variables

We assume that the number of cars is sufficiently large so that it is not necessary to track each one individually. Instead, we describe the system using averaged quantities that vary smoothly in space and time.

10.3 Density

The **density** $\rho(x,t)$ is defined as the number of cars per unit length of road.

Experimental interpretation. To measure ρ at a point $x = x_0$ and time $t = t_0$, we select a small spatial interval

$$x_0 - \Delta x < x < x_0 + \Delta x,$$

and count the number of cars within this interval. Then,

$$\rho(x_0, t_0) \approx \frac{\text{number of cars in the interval at } t = t_0}{2\Delta x}. \tag{5}$$

The parameter Δx must satisfy two competing requirements:

- It should be small enough to capture only cars near x_0.
- It should not be so small that it becomes comparable to the length of individual cars (and their spacing).

In the **continuum limit**, we assume cars are smoothly distributed along the road and define

$$\rho(x_0, t_0) = \lim_{\Delta x \to 0} \frac{\text{number of cars in the interval at } t = t_0}{2\Delta x}.$$

Example (Uniform distribution). Suppose each car has length l and the spacing between consecutive cars is d. Then the number of cars in an interval of length $2\Delta x$ is approximately

$$\frac{2\Delta x}{l + d}.$$

Substituting into (5) and letting $\Delta x \to 0$, we obtain

$$\rho = \frac{1}{l + d}.$$

Maximum density. Since $d \geq 0$, the density satisfies

$$0 < \rho \leq \rho_M, \qquad \text{where } \rho_M = \frac{1}{l}.$$

This corresponds to bumper-to-bumper traffic ($d = 0$).

Merge density. A useful reference value is the **merge density** ρ_{mg}, defined as the density when exactly one car fits between two consecutive cars, that is, when $d = l$. In this case,

$$\rho_{\text{mg}} = \frac{1}{2l}.$$

10.4 Flux

The **flux** $J(x, t)$ represents the rate at which cars pass a point. It has units of cars per unit time.

Experimental interpretation. To measure J at position $x = x_0$ and time $t = t_0$, we select a small time interval

$$t_0 - \Delta t < t < t_0 + \Delta t,$$

and count the net number of cars passing x_0 during this period. We adopt the convention:

- cars moving to the right contribute $+1$,
- cars moving to the left contribute -1.

Then,

$$J(x_0, t_0) \approx \frac{\text{net number of cars passing } x_0}{2\Delta t}. \tag{6}$$

As with density, Δt must be:

- small enough to capture behaviour near t_0,
- but not so small that no cars pass during the interval.

In the **continuum limit**, we define

$$J(x_0, t_0) = \lim_{\Delta t \to 0} \frac{\text{net number of cars passing } x_0 \text{ at } t = t_0}{2\Delta t}.$$

Example (Uniform flow). Return to the example of uniformly distributed cars with length l and spacing d. Suppose each car moves at speed v. In this case, the cars that start out a distance $2\Delta t v$ to the left of x_0 will pass x_0 during the time interval. The number of such cars is approximately

$$\frac{2\Delta t v}{l + d}.$$

Substituting into (6) and letting $\Delta t \to 0$, we obtain

$$J = \frac{v}{l + d} = v\rho.$$

Note: The flux J depends on both the density ρ and the speed v.

10.5 Balance law

We now relate the density $\rho(x, t)$ and the flux $J(x, t)$ through a **balance law**.

The key idea is to track how the number of cars in a fixed spatial interval changes over time.

Assumption. We assume that the number of cars in the interval can only change due to cars entering or leaving through its endpoints. That is:

- cars do not appear or disappear within the interval,
- there are no on-ramps or off-ramps.

Balance principle. Consider the highway interval

$$x_0 - \Delta x < x < x_0 + \Delta x,$$

and a time window around $t = t_0$, that is,

$$t_0 - \Delta t < t < t_0 + \Delta t.$$

Then,

$$
\begin{aligned}
&\text{(number of cars in the highway interval at } t = t_0 + \Delta t\text{)} \\
&\quad - \text{(number of cars in the highway interval at } t = t_0 - \Delta t\text{)} \\
&= \text{(number of cars entering through the left boundary in the time window)} \\
&\quad - \text{(number of cars leaving through the right boundary in the time window).}
\end{aligned}
$$

More precisely, in the continuum limit, using the equations of density and flux, we can write this as

$$2\Delta x \left[\rho(x_0, t_0 + \Delta t) - \rho(x_0, t_0 - \Delta t)\right] = 2\Delta t \left[J(x_0 - \Delta x, t_0) - J(x_0 + \Delta x, t_0)\right].$$

Interpretation.

- If more cars enter than leave, the number of cars in the interval increases.
- If more cars leave than enter, the number decreases.
- If both are equal, the number of cars remains constant.

This statement is the **conservation of cars** and forms the foundation for deriving the traffic flow equation.

10.6 Derivation of the traffic flow equation

We now pass to the **continuum viewpoint**. Using Taylor's theorem (in two variables), we expand ρ and J about (x_0, t_0).

Taylor expansions. For the density, we have

$$\rho(x_0, t_0 + \Delta t) = \rho + \Delta t\, \rho_t + \tfrac{1}{2}(\Delta t)^2 \rho_{tt} + \tfrac{1}{6}(\Delta t)^3 \rho_{ttt} + \cdots,$$
$$\rho(x_0, t_0 - \Delta t) = \rho - \Delta t\, \rho_t + \tfrac{1}{2}(\Delta t)^2 \rho_{tt} - \tfrac{1}{6}(\Delta t)^3 \rho_{ttt} + \cdots.$$

Similarly, for the flux,

$$J(x_0 - \Delta x, t_0) = J - \Delta x\, J_x + \tfrac{1}{2}(\Delta x)^2 J_{xx} - \tfrac{1}{6}(\Delta x)^3 J_{xxx} + \cdots,$$
$$J(x_0 + \Delta x, t_0) = J + \Delta x\, J_x + \tfrac{1}{2}(\Delta x)^2 J_{xx} + \tfrac{1}{6}(\Delta x)^3 J_{xxx} + \cdots.$$

Substituting these into the balance law gives

$$\begin{aligned}
2\Delta x\Big[&(\rho + \Delta t\, \rho_t + \tfrac{1}{2}(\Delta t)^2 \rho_{tt} + \cdots) \\
&- (\rho - \Delta t\, \rho_t + \tfrac{1}{2}(\Delta t)^2 \rho_{tt} + \cdots)\Big] \\
= 2\Delta t\Big[&(J - \Delta x\, J_x + \tfrac{1}{2}(\Delta x)^2 J_{xx} + \cdots) \\
&- (J + \Delta x\, J_x + \tfrac{1}{2}(\Delta x)^2 J_{xx} + \cdots)\Big].
\end{aligned}$$

Here, all quantities ρ and J (and their derivatives) are evaluated at (x_0, t_0).

Leading-order balance. Collecting terms, we obtain

$$\rho_t + O(\Delta t^2) = -J_x + O(\Delta x^2).$$

Letting $\Delta t \to 0$ and $\Delta x \to 0$, we get at the **traffic flow equation**

$$\boxed{\rho_t + J_x = 0.}$$

This is a first-order partial differential equation describing how the density of cars evolves in space and time.

10.7 Velocity and flux

It is often useful to express the flux in terms of the (average) velocity v. Under suitable assumptions, we write

$$J = v\rho.$$

Substituting into the conservation law gives

$$\boxed{\rho_t + (v\rho)_x = 0.}$$

To solve this equation, we typically prescribe an initial condition

$$\rho(x, 0) = f(x),$$

for a given function f.

This model is known as the **Lighthill–Whitham–Richards (LWR) model** of traffic flow.

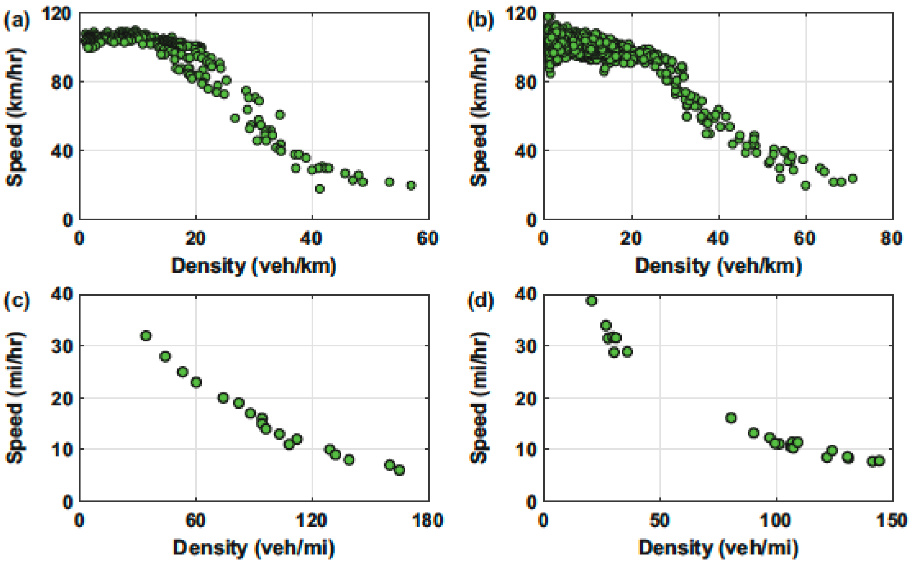


Fig. 5.6 The velocity as a function of the density as measured for different roadways. Shown is (**a**) a highway near Toronto, (**b**) a freeway near Amsterdam, (**c**) the Lincoln Tunnel, and (**d**) the Merritt Parkway. Data for (**a**) and (**b**) are from Rakha and Aerde (2010), and (**c**) and (**d**) are from Greenberg (1959)

Figure 14: Velocity as a function of density for different roadways.

10.8 Constitutive laws

To close the model, we must specify how the velocity v depends on the density ρ. This relationship is called a **constitutive law**.

Qualitative behaviour.

- When ρ is small (light traffic), $v \approx a$ (free-flow speed).
- As ρ increases, v decreases due to interactions between cars.
- As $\rho \to \rho_M$ (maximum density), $v \to 0$.

Hence, $v(\rho)$ is a decreasing function, starting at a when $\rho = 0$ and tending to zero as $\rho \to \rho_M$.

Different choices of $v(\rho)$ lead to different traffic flow models.

10.9 Constant velocity

We begin with the simplest constitutive law:

$$v(\rho) = a,$$

where $a > 0$ is constant.

This assumes cars do not interact and always travel at free-flow speed. While unrealistic, it provides a useful starting point.

In this case,

$$J = a\rho,$$

and the traffic flow equation becomes

$$\rho_t + a\rho_x = 0.$$

This is a linear first-order PDE, which can be solved using the method of characteristics.

Key Idea

This is a **linear, first-order PDE**. It has one unknown function $\rho(x,t)$ and involves its partial derivatives $\rho_t = \partial\rho/\partial t$ and $\rho_x = \partial\rho/\partial x$. Our goal is to find $\rho(x,t)$ given some initial condition $\rho(x,0) = f(x)$.

11 A First Observation: What Does the Equation Say?

Before jumping into a solution method, let us think about what the PDE is telling us geometrically.

Recall the **directional derivative**. If we move in the direction of a vector (a,b) in the (x,t)-plane, the rate of change of ρ in that direction is

$$a\,\rho_x + b\,\rho_t.$$

Our equation $\rho_t + a\,\rho_x = 0$ can be written as

$$a\,\rho_x + 1 \cdot \rho_t = 0,$$

which says: *the directional derivative of ρ in the direction $(a,1)$ is zero.*

Intuition

Along the direction $(a,1)$ in the (x,t)-plane, ρ does not change. In other words, ρ is **constant** along lines that are parallel to the vector $(a,1)$.

This observation is the heart of the method of characteristics.

12 The Method of Characteristics: Building the Idea Step by Step

12.1 Step 1 — Introduce Characteristic Curves

We define special curves in the (x,t)-plane called **characteristics**. A characteristic is a curve along which the PDE reduces to an ODE.

For our equation, we define a characteristic starting at the point $(x_0, 0)$ as the curve

$$x(t) = x_0 + a\,t.$$

This is a straight line with slope $dx/dt = a$ in the (x,t)-plane (or equivalently, slope $1/a$ if you draw t on the vertical axis).

12.2 Step 2 — Show That ρ Is Constant Along Characteristics

Take any point on the characteristic, i.e., let $(x(t), t)$ be a point on the curve $x = x_0 + at$. Define

$$R(t) = \rho\big(x(t),\, t\big) = \rho(x_0 + a\,t,\, t).$$

We compute dR/dt using the **chain rule**:

$$\frac{dR}{dt} = \frac{\partial \rho}{\partial x}\frac{dx}{dt} + \frac{\partial \rho}{\partial t}\frac{dt}{dt} = \rho_x \cdot a + \rho_t \cdot 1.$$

But the PDE says $\rho_t + a\,\rho_x = 0$, so:

$$\boxed{\frac{dR}{dt} = \rho_t + a\,\rho_x = 0.}$$

Key Idea

Along every characteristic $x = x_0 + a\,t$, the density ρ satisfies the ODE

$$\frac{dR}{dt} = 0,$$

which means $R(t) = \text{constant}$. That is:

$$\rho(x_0 + a\,t,\ t) = \rho(x_0,\, 0) = f(x_0).$$

12.3 Step 3 — Write the General Solution

Given any point (x,t) in the domain, we ask: *which characteristic passes through it?* The characteristic through $(x_0, 0)$ reaches (x,t) when

$$x = x_0 + a\,t \quad \Longrightarrow \quad x_0 = x - a\,t.$$

Since ρ is constant along this characteristic,

$$\rho(x,\, t) = f(x_0) = f(x - a\,t).$$

General Solution

The solution to the initial value problem

$$\rho_t + a\,\rho_x = 0, \qquad \rho(x,0) = f(x)$$

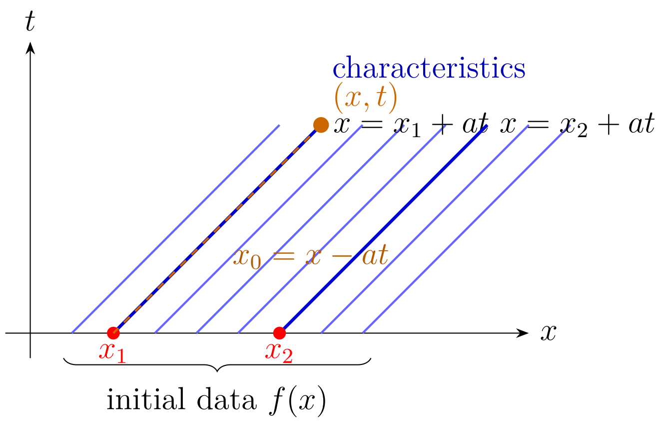
is

$$\boxed{\rho(x,\,t) = f(x - a\,t).}$$

The initial profile f is **transported** (shifted) to the right at speed a without any change in shape.

13 Geometric Picture: The Characteristic Diagram

It is essential to visualize the characteristics. In the (x,t)-plane (with $t \geq 0$), each characteristic is a straight line of slope $1/a$ (or speed a if we put x on the horizontal axis).



Each characteristic carries the value of the initial data at its foot $x_0 = x - at$ forward in time. To find ρ at any point (x,t), just trace back along the characteristic to the x-axis and read off ρ_0.

14 Worked Examples

Example:

Smooth initial data — a Gaussian bump
Suppose $a = 1$ and

$$f(x) = e^{-x^2}.$$

By our formula, the solution is

$$\rho(x,t) = e^{-(x-t)^2}.$$

This is a Gaussian centred at $x = t$, moving to the right at speed 1. The bump travels without distortion. At $t = 0$ it sits at $x = 0$; at $t = 2$ it sits at $x = 2$, and so on.

Verification. Let us check directly:

$$\rho_t = 2(x-t)\, e^{-(x-t)^2}, \qquad \rho_x = -2(x-t)\, e^{-(x-t)^2}.$$

Then

$$\rho_t + 1 \cdot \rho_x = 2(x-t)e^{-(x-t)^2} - 2(x-t)e^{-(x-t)^2} = 0.\checkmark$$

Example:

Piecewise constant data — a traffic jam front
Suppose $a = 30$ (km/h) and

$$f(x) = \begin{cases} \rho_H & x < 0 \quad \text{(high density: congested region)}, \\ \rho_L & x \geq 0 \quad \text{(low density: free road ahead)}, \end{cases}$$

with $\rho_H > \rho_L$. The solution is

$$\rho(x,t) = f(x - 30t) = \begin{cases} \rho_H & x - 30t < 0, \\ \rho_L & x - 30t \geq 0. \end{cases}$$

The discontinuity (the "front" between congested and free traffic) moves to the right at speed 30 km/h. The entire density profile—including the jump—is simply advected at speed a. No new information is created; the shape is frozen in time.

Physical interpretation. At time t, the front has moved to position $x = 30t$. Every car, whether in the congested region or not, moves at the same speed $a = 30$ km/h, so the pattern just shifts rigidly.

Example:

Sinusoidal initial data
Let $a = 2$ and $f(x) = 1 + \sin(x)$. Then

$$\rho(x,t) = 1 + \sin(x - 2t).$$

This is a sinusoidal wave travelling to the right at speed 2. The wave number, amplitude, and shape are all preserved; only the phase changes.

15 Lecture 16

15.1 Summary of the Method of Characteristics

15.2 Recap: Method of Characteristics (Linear Case)

We briefly review the method of characteristics for the linear equation.

Problem:

$$\rho_t + a\,\rho_x = 0, \qquad \rho(x,0) = f(x).$$

Key idea. We look for curves in the (x,t)-plane along which the PDE reduces to an ODE.

Procedure.

1. Solve the characteristic equation:
$$\frac{dx}{dt} = a \quad \Rightarrow \quad x = x_0 + at.$$
2. Along these curves,
$$\frac{d}{dt}\rho(x(t),t) = 0 \quad \Rightarrow \quad \rho = \text{constant}.$$
3. Using the initial condition,
$$\rho = f(x_0).$$
4. Since $x_0 = x - at$, the solution is
$$\rho(x,t) = f(x - at).$$

Key Idea

For the linear equation, characteristics are straight lines with constant slope a, and the solution is transported along them.

Finite Length Highways:

Solve the equation
$$\rho_t + 4\rho_x = 0$$
on the domain $0 < x < 1$ and $t > 0$, with the initial condition $\rho(x,0) = 2$, and the boundary condition $\rho(0,t) = 2e^{-t}$.

Solution. The characteristic curves are given by $x = x_0 + 4t$. Along each characteristic, ρ is constant. The initial condition $\rho(x,0) = 2$ implies that $\rho = 2$ along the characteristics that originate from the x-axis. The boundary condition $\rho(0,t) = 2e^{-t}$ implies that $\rho = 2e^{-t}$ along the characteristics that originate from the t-axis.

15.3 Why Is This the Only Solution? (Uniqueness, Informally)

Every point (x, t) in the upper half-plane $t > 0$ lies on exactly one characteristic, namely the line $x = (x - at) + at$ rooted at $x_0 = x - at$ on the x-axis. The value of ρ at that root is prescribed by the initial condition. Since ρ is constant along the characteristic, $\rho(x, t)$ is completely determined. There is no freedom left — the solution is unique.

15.4 From Linear to Nonlinear Traffic Flow

Recall the traffic flow equation

$$\rho_t + J_x = 0, \qquad J = v\rho.$$

Using the chain rule,

$$J_x = \frac{dJ}{d\rho}\,\rho_x = J_\rho\,\rho_x.$$

Thus, the equation becomes

$$\rho_t + J_\rho\,\rho_x = 0.$$

Linear case. If $J = a\rho$, then $J_\rho = a$ is constant, and we recover the linear equation.

Nonlinear case. We now allow J to depend nonlinearly on ρ:

$$J = \rho F(\rho).$$

Define

$$c(\rho) := J_\rho = F(\rho) + \rho F'(\rho),$$

called the **wave velocity**. The PDE becomes

$$\rho_t + c(\rho)\,\rho_x = 0.$$

Remarks: This PDE is very general, and a couple of restrictions are needed to help guarantee that there is a solution. One is that $c(\rho)$ should be a smooth function of ρ. Second is that $c'(\rho) \neq 0$ (otherwise the equation becomes degenerate, and we need to use a different method to solve it).

15.5 Greenshields Law

We now consider a specific constitutive law.

Greenshields law:

$$v(\rho) = v_{\mathrm{M}}\left(1 - \frac{\rho}{\rho_M}\right),$$

where:

- v_{M} is the free-flow speed,
- ρ_M is the maximum density.

Then the flux is

$$J(\rho) = v(\rho)\rho = v_{\mathrm{M}}\left(\rho - \frac{\rho^2}{\rho_M}\right),$$

and hence

$$c(\rho) = J_\rho = v_{\mathrm{M}}\left(1 - \frac{2\rho}{\rho_M}\right).$$

Interpretation.

- $c(\rho) > 0$ for light traffic ($\rho < \rho_{\mathrm{mg}}$),
- $c(\rho) = 0$ at $\rho = \rho_{\mathrm{mg}}$,
- $c(\rho) < 0$ for heavy traffic ($\rho > \rho_{\mathrm{mg}}$).

Note that $v(\rho) \geq 0$, but $c(\rho)$ can be negative. This means that *information* (traffic waves) can travel backwards even though cars move forwards.

15.6 Method of Characteristics (Nonlinear Case)

We now apply the method of characteristics to

$$\rho_t + c(\rho)\rho_x = 0.$$

We again look for curves $x = X(t)$ such that ρ is constant along them:

$$\frac{d}{dt}\rho(X(t), t) = 0.$$

Using the chain rule,

$$\frac{d}{dt}\rho(X(t), t) = \rho_t + X'(t)\rho_x.$$

Thus, choosing

$$X'(t) = c(\rho),$$

gives

$$\frac{d}{dt}\rho = 0.$$

Key consequence.

- ρ is constant along characteristics,
- but the slope of the characteristics depends on ρ.

If $\rho_0 = f(x_0)$, then the characteristics satisfy

$$x = x_0 + c(\rho_0)t.$$

Key Idea

In the nonlinear case, characteristics are still straight lines, but their slope depends on the initial data.

15.7 Important Remark

Although the characteristic curves are linear, the equation is nonlinear because:

- the slope depends on ρ,
- different characteristics travel at different speeds,
- this leads to interactions (crossing, shocks, etc.).

15.8 Example: Modified Red Light–Green Light

We consider a smoothed initial condition:

$$f(x) = \begin{cases} \rho_H, & x < -\varepsilon, \\ \rho_H + \dfrac{\rho_L - \rho_H}{2\varepsilon}(x + \varepsilon), & -\varepsilon \leq x < \varepsilon, \\ \rho_L, & x \geq \varepsilon, \end{cases}$$

with $\rho_H > \rho_L$.

Using Greenshields law,

$$c(\rho) = v_{\mathrm{M}} \left(1 - \frac{2\rho}{\rho_M}\right).$$

Characteristics:

- If $x_0 < -\varepsilon$: $\rho_0 = \rho_H$,
$$x = x_0 + c(\rho_H)t.$$
- If $-\varepsilon \leq x_0 < \varepsilon$: $\rho_0 = f(x_0)$,
$$x = x_0 + c(f(x_0))t.$$
- If $x_0 \geq \varepsilon$: $\rho_0 = \rho_L$,
$$x = x_0 + c(\rho_L)t.$$

Task. Find the full solution $\rho(x, t)$ by expressing x_0 in terms of (x, t).

Modified red light–green light:

Consider
$$\rho_t + c(\rho)\rho_x = 0, \qquad c(\rho) = v_{\mathrm{M}}\left(1 - \frac{2\rho}{\rho_M}\right),$$
with initial condition
$$f(x) = \begin{cases} \rho_H, & x < -\varepsilon, \\ \rho_H + \dfrac{\rho_L - \rho_H}{2\varepsilon}(x+\varepsilon), & -\varepsilon \le x < \varepsilon, \\ \rho_L, & x \ge \varepsilon, \end{cases} \qquad \rho_H > \rho_L.$$

We want to find the full solution $\rho(x,t)$.

Step 1: Characteristics.
If x_0 is the starting point of a characteristic on the x-axis, then
$$\rho_0 = f(x_0),$$
and the characteristic is
$$x = x_0 + c(\rho_0)t.$$
Along this characteristic, the solution is constant:
$$\rho(x,t) = \rho_0 = f(x_0).$$

Step 2: Regions of the solution.

- If $x_0 < -\varepsilon$, then $f(x_0) = \rho_H$, so the characteristics are
$$x = x_0 + c(\rho_H)t.$$
Hence the left constant state occupies the region
$$x < -\varepsilon + c(\rho_H)t,$$
and there
$$\rho(x,t) = \rho_H.$$
- If $x_0 \ge \varepsilon$, then $f(x_0) = \rho_L$, so the characteristics are
$$x = x_0 + c(\rho_L)t.$$
Hence the right constant state occupies the region
$$x > \varepsilon + c(\rho_L)t,$$
and there
$$\rho(x,t) = \rho_L.$$

- If $-\varepsilon \le x_0 < \varepsilon$, then

$$f(x_0) = \rho_H + \frac{\rho_L - \rho_H}{2\varepsilon}(x_0 + \varepsilon).$$

In this region the characteristic is

$$x = x_0 + c(f(x_0))\,t.$$

We now solve this relation for x_0.

Step 3: Solve for x_0 in the transition region.
Since $c(\rho)$ is linear in ρ, and $f(x_0)$ is linear in x_0, the quantity $c(f(x_0))$ is also linear in x_0. In fact,

$$c(f(x_0)) = c(\rho_H) + \frac{c(\rho_L) - c(\rho_H)}{2\varepsilon}(x_0 + \varepsilon).$$

Therefore,

$$x = x_0 + \left[c(\rho_H) + \frac{c(\rho_L) - c(\rho_H)}{2\varepsilon}(x_0 + \varepsilon)\right] t.$$

Rearranging gives

$$x_0 = \frac{x - \frac{1}{2}\big(c(\rho_H) + c(\rho_L)\big)t}{1 + \frac{c(\rho_L) - c(\rho_H)}{2\varepsilon}t}.$$

Substituting this into $\rho = f(x_0)$ yields the solution in the transition region:

$$\rho(x,t) = \rho_H + \frac{\rho_L - \rho_H}{2\varepsilon}\left(\frac{x + \varepsilon - c(\rho_H)t}{1 + \frac{c(\rho_L) - c(\rho_H)}{2\varepsilon}t}\right).$$

Hence, the full solution is

$$\rho(x,t) = \begin{cases} \rho_H, & x < -\varepsilon + c(\rho_H)t, \\ \rho_H + \dfrac{\rho_L - \rho_H}{2\varepsilon}\left(\dfrac{x + \varepsilon - c(\rho_H)t}{1 + \frac{c(\rho_L) - c(\rho_H)}{2\varepsilon}t}\right), & -\varepsilon + c(\rho_H)t \le x < \varepsilon + c(\rho_L)t, \\ \rho_L, & x \ge \varepsilon + c(\rho_L)t. \end{cases}$$

Here

$$c(\rho_H) = v_{\mathrm{M}}\left(1 - \frac{2\rho_H}{\rho_M}\right), \qquad c(\rho_L) = v_{\mathrm{M}}\left(1 - \frac{2\rho_L}{\rho_M}\right).$$

Remark. Since $\rho_H > \rho_L$, we have

$$c(\rho_H) < c(\rho_L).$$

Thus the characteristics spread out rather than intersect, so this produces a smooth expanding transition region.

16 Shock Waves

As we saw in the previous lecture, when $\rho_L < \rho_R$ (faster cars behind, slower cars ahead), the characteristics from the left travel faster than those from the right. They eventually intersect, and the smooth solution can no longer be continued.

At the moment of intersection, the solution develops a *jump discontinuity* that moves along the x-axis. This moving jump is called a **shock wave**.

Question. How fast does the shock move?

16.1 Setting Up the Problem

Suppose the solution has a jump located at $x = s(t)$, with

$$\rho(x,t) = \begin{cases} \rho_L, & x < s(t), \\ \rho_R, & x > s(t). \end{cases}$$

Since ρ_x is not defined at $x = s(t)$, the PDE $\rho_t + J_x = 0$ cannot be applied directly there. Instead, we integrate over a small interval $[s-\varepsilon,\ s+\varepsilon]$ straddling the jump.

16.2 Deriving the Rankine–Hugoniot Condition

Step 1: Integrate the conservation law.
Integrating $\rho_t + J_x = 0$ over $[s-\varepsilon, s+\varepsilon]$ gives

$$\int_{s-\varepsilon}^{s+\varepsilon} \rho_t \, dx + J(\rho_R) - J(\rho_L) = 0.$$

Step 2: Differentiate the total car count.
By the Fundamental Theorem of Calculus,

$$\frac{d}{dt}\int_{s-\varepsilon}^{s+\varepsilon} \rho \, dx = \int_{s-\varepsilon}^{s+\varepsilon} \rho_t \, dx + s'(t)\,\rho\big|_{x=s+\varepsilon} - s'(t)\,\rho\big|_{x=s-\varepsilon}.$$

Step 3: Use the piecewise–constant structure.
Since $\rho = \rho_L$ on $[s-\varepsilon, s]$ and $\rho = \rho_R$ on $[s, s+\varepsilon]$,

$$\int_{s-\varepsilon}^{s+\varepsilon} \rho \, dx = \varepsilon(\rho_L + \rho_R),$$

whose time derivative is zero (the densities are constants). Therefore,

$$\int_{s-\varepsilon}^{s+\varepsilon} \rho_t \, dx = s'(t)\rho_R - s'(t)\rho_L = s'(t)(\rho_R - \rho_L).$$

Step 4: Combine.
Substituting into Step 1,

$$s'(t)(\rho_R - \rho_L) + J(\rho_R) - J(\rho_L) = 0,$$

and rearranging gives

Key Idea

Rankine–Hugoniot condition (flux version)

$$s'(t) = \frac{J(\rho_R) - J(\rho_L)}{\rho_R - \rho_L}.$$

This determines the velocity of the shock.

16.3 Intuition: a Car-Counting Argument

There is a cleaner way to see where the formula comes from, without any calculus.

Watch a fixed stretch of road $[A, B]$ containing the shock at $x = s(t)$. Cars are conserved, so the only way the total number of cars in $[A, B]$ changes is through cars entering at A and leaving at B:

$$\frac{d}{dt}(\text{cars in } [A, B]) = J(\rho_L) - J(\rho_R).$$

Now count directly. The shock splits $[A, B]$ into two pieces:

$$\text{cars in } [A, B] = \underbrace{\rho_L\,(s - A)}_{\text{left piece}} + \underbrace{\rho_R\,(B - s)}_{\text{right piece}}.$$

Differentiating with respect to t (only s moves):

$$\frac{d}{dt}(\text{cars in } [A, B]) = \rho_L\, s'(t) - \rho_R\, s'(t) = (\rho_L - \rho_R)\, s'(t).$$

Setting the two expressions equal and solving for $s'(t)$ recovers the Rankine–Hugoniot condition exactly.

Physical interpretation. Suppose the shock moves rightward by a small amount ds in time dt. The strip of road $[s,\, s + ds]$ has just switched from density ρ_R to density ρ_L. The number of cars in that strip changed from $\rho_R\, ds$ to $\rho_L\, ds$, a loss of

$$(\rho_R - \rho_L)\, ds \quad \text{cars.}$$

These cars must be accounted for by the net flux across the shock. Over time dt, the net outflow is $(J(\rho_R) - J(\rho_L))\, dt$. Conservation requires

$$(\rho_R - \rho_L)\, ds = (J(\rho_R) - J(\rho_L))\, dt,$$

and dividing by dt gives the Rankine–Hugoniot condition directly.

16.4 Wave Velocity Version

It is useful to rewrite the condition in terms of the wave speed $c(\rho) = J'(\rho)$. Since $J(0) = 0$, we have

$$J(\rho) = \int_0^{\rho} c(r)\, dr,$$

so $J(\rho_R) - J(\rho_L) = \int_{\rho_L}^{\rho_R} c(\rho)\, d\rho$. Therefore,

Key Idea

Rankine–Hugoniot condition (wave velocity version)

$$s'(t) = \frac{1}{\rho_R - \rho_L} \int_{\rho_L}^{\rho_R} c(\rho)\, d\rho.$$

The shock travels at the *average wave speed* over all density values between ρ_L and ρ_R.

16.5 Two Types of Jump

Not every jump in ρ is a shock. The distinction depends on the car velocity v:

- **Contact discontinuity.** ρ has a jump at $x = s(t)$, but v is continuous there. The red-light/green-light solution (linear case) produces contact discontinuities: the velocity is the constant a on both sides, so the Rankine–Hugoniot condition gives $s'(t) = a$. The jump is simply advected at the flow speed.
- **Shock.** Both ρ and v are discontinuous at $x = s(t)$. The motion of the shock is governed by the Rankine–Hugoniot condition and depends on the constitutive law.

16.6 Example: Greenshields Law

Under Greenshields law $v(\rho) = v_M(1 - \rho/\rho_M)$, the wave speed is

$$c(\rho) = v_M \left(1 - \frac{2\rho}{\rho_M}\right),$$

which is linear in ρ. The integral in the wave velocity version is therefore trivial:

$$s'(t) = \frac{1}{\rho_R - \rho_L} \int_{\rho_L}^{\rho_R} v_M \left(1 - \frac{2\rho}{\rho_M}\right) d\rho = \frac{1}{2}\big(c(\rho_R) + c(\rho_L)\big) = \frac{1}{2}(c_R + c_L).$$

Key Idea

Under Greenshields law, the shock speed equals the average of the wave speeds on the two sides of the shock.

Faster cars behind: shock formation and speed:

Consider the piecewise–constant initial condition

$$\rho(x, 0) = \begin{cases} \rho_L = 1, & x < 0, \\ \rho_R = 4, & x > 0, \end{cases}$$

with Greenshields parameters $v_M = 3$, $\rho_M = 6$.

Step 1: Compute the wave speeds.

$$c(\rho_L) = 3\left(1 - \frac{2}{6}\right) = 3 \cdot \frac{2}{3} = 2, \qquad c(\rho_R) = 3\left(1 - \frac{8}{6}\right) = 3 \cdot \left(-\frac{1}{3}\right) = -1.$$

Since $c(\rho_L) = 2 > 0$ and $c(\rho_R) = -1 < 0$, characteristics from both sides move *toward* $x = 0$. They collide immediately, and a shock is present for all $t > 0$.

Step 2: Apply the Rankine–Hugoniot condition.

$$s'(t) = \tfrac{1}{2}(c_R + c_L) = \tfrac{1}{2}(-1 + 2) = \tfrac{1}{2}.$$

Step 3: Integrate.
Since $s'(t) = \frac{1}{2}$ is constant and $s(0) = 0$,

$$s(t) = \tfrac{1}{2}\, t.$$

The shock moves to the right at speed $\frac{1}{2}$.

Hence, the full solution is

$$\rho(x,t) = \begin{cases} 1, & x < \frac{t}{2}, \\ 4, & x > \frac{t}{2}. \end{cases}$$

Remark. Cars on the left travel at $v(\rho_L) = 3(1 - 1/6) = 5/2$, which is faster than the shock speed $1/2$. Cars from behind therefore *catch up* to the shock and join the dense pack. This is consistent with the physical picture: the back of a traffic jam slowly drifts forward, but the cars behind are running into it.

16.7 Three Velocities

We now have three distinct velocities in the problem.

- $v(x,t)$: the velocity of the cars at position x.
- $c(\rho) = J'(\rho)$: the wave velocity, giving the slope of characteristics.
- $s'(t)$: the shock velocity, given by the Rankine–Hugoniot condition.

In a nonlinear problem these three are genuinely independent: they are not simple multiples of one another. A single velocity is not sufficient to describe the full dynamics of the solution.

17 Appendix A: Taylor Expansions

The following list collects common Taylor (Maclaurin) expansions about $x = 0$, written with explicit remainder terms in Big–O notation.

Polynomials and Rational Functions

$$\begin{aligned}
\frac{1}{1-x} &= 1 + x + x^2 + x^3 + O(x^4), \\
\frac{1}{(1-x)^2} &= 1 + 2x + 3x^2 + 4x^3 + O(x^4), \\
\frac{1}{1+x} &= 1 - x + x^2 - x^3 + O(x^4), \\
(1+x)^\alpha &= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{6}x^3 + O(x^4), \\
(a+x)^\gamma &= a^\gamma + \gamma a^{\gamma-1}x + \frac{\gamma(\gamma-1)}{2}a^{\gamma-2}x^2 + \frac{\gamma(\gamma-1)(\gamma-2)}{6}a^{\gamma-3}x^3 + O(x^4).
\end{aligned}$$

Exponential and Logarithmic Functions

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + O(x^4), \\
e^{-x} &= 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + O(x^4), \\
a^x &= 1 + (\ln a)x + \frac{(\ln a)^2}{2}x^2 + \frac{(\ln a)^3}{6}x^3 + O(x^4), \\
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + O(x^5), \\
\ln(1-x) &= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + O(x^5), \\
\ln(a+x) &= \ln a + \frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} + O(x^4).
\end{aligned}$$

Trigonometric Functions

$$\begin{aligned}
\sin x &= x - \frac{x^3}{6} + \frac{x^5}{120} + O(x^7), \\
\cos x &= 1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6), \\
\tan x &= x + \frac{x^3}{3} + \frac{2x^5}{15} + O(x^7), \\
\cot x &= \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} + O(x^5).
\end{aligned}$$

Remark:

The expansion for $\cot x$ is a Laurent expansion (since $\cot x$ has a pole at $x = 0$), not a Taylor series.

Shifted Trigonometric Functions

$$\begin{aligned}
\sin(a + x) &= \sin a + x \cos a - \frac{x^2}{2} \sin a - \frac{x^3}{6} \cos a + O(x^4), \\
\cos(a + x) &= \cos a - x \sin a - \frac{x^2}{2} \cos a + \frac{x^3}{6} \sin a + O(x^4), \\
\tan(a + x) &= \tan a + x \sec^2 a + x^2 \tan a \sec^2 a + O(x^3).
\end{aligned}$$

Inverse Trigonometric Functions

$$\begin{aligned}
\arctan x &= x - \frac{x^3}{3} + \frac{x^5}{5} + O(x^7), \\
\operatorname{arccot} x &= \frac{\pi}{2} - x + \frac{x^3}{3} - \frac{x^5}{5} + O(x^7), \\
\arcsin x &= x + \frac{x^3}{6} + \frac{3x^5}{40} + O(x^7), \\
\arccos x &= \frac{\pi}{2} - x - \frac{x^3}{6} - \frac{3x^5}{40} + O(x^7).
\end{aligned}$$

Hyperbolic and Inverse Hyperbolic Functions

$$\begin{aligned}
\sinh x &= x + \frac{x^3}{6} + \frac{x^5}{120} + O(x^7), \\
\cosh x &= 1 + \frac{x^2}{2} + \frac{x^4}{24} + O(x^6), \\
\tanh x &= x - \frac{x^3}{3} + \frac{2x^5}{15} + O(x^7), \\
\operatorname{arcsinh} x &= x - \frac{x^3}{6} + \frac{3x^5}{40} + O(x^7), \\
\operatorname{arccosh}(1 + x) &= \sqrt{2x} \left(1 - \frac{x}{12} + \frac{3x^2}{160} + O(x^3) \right), \\
\operatorname{arctanh} x &= x + \frac{x^3}{3} + \frac{x^5}{5} + O(x^7).
\end{aligned}$$

Square Roots and Fractional Powers

$$\begin{aligned}
\sqrt{1 + x} &= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + O(x^4), \\
(1 + x)^{-1/2} &= 1 - \frac{x}{2} + \frac{3x^2}{8} - \frac{5x^3}{16} + O(x^4).
\end{aligned}$$

References

[1] M. H. Holmes, *Introduction to the Foundations of Applied Mathematics*, 2nd edn., Springer, 2019.

[2] M. H. Holmes, *Introduction to Perturbation Methods*, 2nd edn., Springer, 2013.